Discrete Cosine Transform

Discrete Cosine transform *(DCT)* is the basis for many image and video compression algorithms, especially the baseline *JPEG* and *MPEG* standards for compression of still and video images respectively.

The one dimensional discrete cosine transform *(1D DCT)* of *N* samples is formulated by:

$$F(u) = \sqrt{\frac{2}{N}} C(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right]$$

for u=0,1,2,.........., N-1 where:

$$C(u) = \begin{cases} \frac{1}{\sqrt{2}} & for \quad u = 0 \\ 1 & otherwise \end{cases}$$

f(x): represents the value of x^{th} sample of input signal.

F(u) : represents a *DCT* coefficient *for u=0,1 , ... , N-1*

The one dimensional inverse discrete cosine transform *(1D IDCT)* is formulated as follows:

$$f(x) = \sqrt{\frac{2}{N}} \sum_{u=0}^{N-1} C(u) F(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right]$$

for x=0,1,2,............,N-1

The *DCT* is based on defining a matrix *[C]* such that:

$$C(k,n) = \frac{1}{\sqrt{N}} \qquad for \;\; k,n = 0$$

$$C(k,n) = \sqrt{\frac{2}{N}} \quad \cos\left[\frac{\pi(2k+1)n}{2N}\right]$$

if an image is divided into blocks of size *8*8* then the matrix *C* must also be *8*8* if a matrix *U* represents the pixels in an *8*8* block of an image, then the *DCT* is define by the equation: $[V]=[C][U][C^T]$

C^T: the transpose of matrix *[C]*

The inverse *DCT* can be used to regenerate an approximation to the matrix *[U]* using the equation: $[U']=[C^T][V][C]$

Where *U'* represent the reconstructed image.

A compression scheme using the DCT can be summarized in the following steps:

a. Divide the image in to *8*8* blocks.
b. Perform the *DCT* on each *8*8* block.
c. Replace values close to zero with zero.
d. Huffman encode the resulting file.

Decompression involves the following

a. Huffman decode the compressed file.
b. Perform the inverse *DCT* on each *8*8* block.

Example: ***Apply the DCT algorithm to the following points: [2 3 4 4]***
Solution:

$$F(u) = \sqrt{\frac{2}{N}} C(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right]$$

$$F(0) = \sqrt{\frac{2}{4}} * \frac{1}{\sqrt{2}} \left[\begin{array}{l} f(0)\cos\frac{\pi(2*0+1)0}{2*4} + f(1)\cos\frac{\pi(2*1+1)0}{2*4} + \\ f(2)\cos\frac{\pi(2*2+1)0}{2*4} + f(3)\cos\frac{\pi(2*3+1)0}{2*4} \end{array}\right]$$

$$= \frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}} [2+3+4+4]$$

$$= \frac{1}{2} * 13$$

= ***6.5***

$$F(1) = \sqrt{\frac{2}{4}} * 1 \left[\begin{array}{l} f(0)\cos\frac{\pi(2*0+1)1}{2*4} + f(1)\cos\frac{\pi(2*1+1)1}{2*4} + \\ f(2)\cos\frac{\pi(2*2+1)1}{2*4} + f(3)\cos\frac{\pi(2*3+1)1}{2*4} \end{array}\right]$$

$$= \frac{1}{\sqrt{2}} \left[2*\cos\frac{\pi}{8} + 3*\cos\frac{3\pi}{8} + 4*\cos\frac{5\pi}{8} + 4*\cos\frac{7\pi}{8}\right]$$

$$= \frac{1}{\sqrt{2}} [\ 2*0.92 + 3*0.38 + 4*-0.38 + 4*-0.92]$$

$$= \frac{1}{\sqrt{2}} [1.84 + 1.14 - 1.52 - 3.68]$$

$$= \frac{1}{\sqrt{2}} [2.98 - 5.2]$$

$$= \frac{1}{\sqrt{2}} [-2.22]$$

= ***- 1.57***

$$F(2) = \sqrt{\frac{2}{4}} * 1 \left[\begin{array}{l} f(0)\cos\frac{\pi(2*0+1)2}{2*4} + f(1)\cos\frac{\pi(2*1+1)2}{2*4} + \\ f(2)\cos\frac{\pi(2*2+1)2}{2*4} + f(3)\cos\frac{\pi(2*3+1)2}{2*4} \end{array}\right]$$

$$= \frac{1}{\sqrt{2}} \left[2*\cos\frac{2\pi}{8} + 3*\cos\frac{6\pi}{8} + 4*\cos\frac{10\pi}{8} + 4*\cos\frac{14\pi}{8}\right]$$

$$= \frac{1}{\sqrt{2}} [\ 2*0.707 + 3*-0.707 + 4*-0.707 + 4*0.707]$$

$$= \frac{1}{\sqrt{2}} [1.414 - 2.121 - 2.828 + 2.828]$$

$= \frac{1}{\sqrt{2}}$ [-0.707]

$= \frac{1}{1.414}[-0.707]$

= 0.707 * -0.707

= ***- 0.5***

$$F(3) = \sqrt{\frac{2}{4}} \;*1 \left[\begin{array}{l} f(0)\cos\frac{\pi(2*0+1)3}{2*4} + f(1)\cos\frac{\pi(2*1+1)3}{2*4} + \\ f(2)\cos\frac{\pi(2*2+1)3}{2*4} + f(3)\cos\frac{\pi(2*3+1)3}{2*4} \end{array} \right]$$

$$= \frac{1}{\sqrt{2}} \left[2*\cos\frac{3\pi}{8} + 3*\cos\frac{9\pi}{8} + 4*\cos\frac{15\pi}{8} + 4*\cos\frac{21\pi}{8} \right]$$

= 0.707 * [2* 0.38 + 3* -0.92 + 4* 0.92 +4 * -0.38]

= 0.707 * [0.76 – 2.76 + 3.68 – 1.52]

= 0.707 * [0.16]

= ***0.11***