Chapter One

الصفات التبولوجية والصفات غير التبولوجية

Topological and Non-Topological Properties

Definition (1.1): (Open and Closed Functions)

We say that the function $f: (X, \tau) \to (X^*, \tau^*)$ is:

(1) **Open** iff $f(G)$ is open in X^* for each open G in X.

(2) **Closed** iff $f(F)$ is closed in X^* for each closed F in X.

Example (1.1): Let (X, τ) be any topological space, and let $X^* = \{a, b, c\}$, $\tau^* = \{\emptyset, \{a\}, \{a, c\}, X^*\}$. Then $f: (X, \tau) \to (X^*, \tau^*)$ such that $f(x) = a$, $\forall\ x \in X$ is

(i) Continuous

(ii) Open

(iii) Not closed

Proof:

(i) The open sets containing $f(x) = a$ are $\{a\}$, $\{a, c\}$, X^*, and any open $G \ni x$,

$f(G) = \{a\}, \forall^{open}\ G^* \ni f(x), \exists^{open}\ G \ni x, such\ that\ f(G) \subset G^*$

$\therefore\ f$ is continuous

(ii) Let G be any open set in X

Now, $f(G) = \{a\}$ is open in X^* $\Rightarrow$ f is open function

(iii) f is not closed

Since $f(F) = \{a\}$ not closed $\forall^{closed}\ F \subset X$

Theorem (1.1): Let $f: (X, \tau) \to (X^*, \tau^*)$ and $E \subset X$. Then f is open iff $f(E^\circ) \subset \left(f(E)\right)^\circ, \forall\ E \subset X$.

Proof: Let f is open and $E^\circ \subset X$ is open

$\Rightarrow\ f(E^\circ)$ is open

We have

$E^\circ \subset E$

$\Rightarrow\ f(E^\circ) \subset f(E)$

$\Rightarrow\ \left(f(E^\circ)\right)^\circ \subset \left(f(E)\right)^\circ$

$\Rightarrow\ f(E^\circ) \subset \left(f(E)\right)^\circ$

Conversely:

Assume that $f(E^\circ) \subset \left(f(E)\right)^\circ,\ \forall\, E \subset X$

Let $G \subset X$ be an open set

We get

$f(G^\circ) \subset \left(f(G)\right)^\circ$

$\Rightarrow\ f(G) \subset \left(f(G)\right)^\circ$ (G is open set)

But $\left(f(G)\right)^\circ \subset f(G)$

$\Rightarrow\ f(G) = \left(f(G)\right)^\circ$

$\Rightarrow\ f(G)$ is open set

$\Rightarrow\ f$ is open function

Theorem (1.2): Let $f: (X, \tau) \to (X^*, \tau^*)$ and $E \subset X$. Then f is closed iff $\overline{(f(E))} \subset f(\bar{E}), \forall\, E \subset X$.

Proof: We have f closed and $\bar{E}$ is closed

$\Rightarrow\ f(\bar{E})$ closed

We have $E \subset \bar{E}$

$\Rightarrow\ f(E) \subset f(\bar{E})$

$\Rightarrow\ \overline{f(E)} \subset \overline{f(\bar{E})}$

$\Rightarrow\ \overline{f(E)} \subset f(\bar{E}), \forall\, E \subset X$

Conversely:

Assume that $\overline{f(E)} \subset f(\bar{E}), \forall\, E \subset X$

Let $F \subset X$ be closed set (i.e. $F = \bar{F}$)

We get

$\overline{f(F)} \subset f(\bar{F})$

$\Rightarrow\ \overline{f(F)} \subset f(F)$ (F is closed set)

But $f(F) \subset \overline{f(F)}$

$\Rightarrow\ f(F) = \overline{f(F)}$

$\Rightarrow\ f(F)$ closed set

$\Rightarrow\ f$ is closed function

Exercises (1.1): (Homework)

(1) If $g: (X, \tau) \to (X^*, \tau^*)$ is defined by $g(x) = b, \forall\, x \in X$. Let (X, τ) be any topological space, and $X^* = \{a, b, c\}$, $\tau^* = \{\emptyset, \{a\}, \{a, c\}, X^*\}$. Discuss the properties of g continuous, open and closed.

Definition (1.2): (Topological Homeomorphism) التـشـاكـل التبـولـوجـي

We say that the function $f: (X, \tau) \to (X^*, \tau^*)$ is a **topological homeomorphism** iff the following conditions satisfied:

(3) f is continuous.

(4) f is one-to-one.

(5) f is onto.

(6) f is open.

Remark (1.1): According to the Fundamental Theorem of Continuity, the condition (f is open) can be replace by the condition (f^{-1} is continuous).

Example (1.2): Let (X, τ) be any topological space, and let $X^* = \{a, b, c\}$, $\tau^* = \{\emptyset, \{a\}, \{a, c\}, X^*\}$. Define $f: (X, \tau) \to (X^*, \tau^*)$ as follows: $f(x) = a$, $\forall\, x \in X$. Determine whether f is a homeomorphism is or not.