**Theorem (4.2):** Every $T_3$-space is $T_2$-space.

**Proof:** Let $(X,\tau)$ be $T_3$-space

We need to show that $(X,\tau)$ is $T_2$-space

Let $x, y \in X, x \neq y$

Since $(X,\tau)$ is $T_1$-space (since $T_3$-space)

$\Rightarrow$ $\{y\}$ is closed in $X$

$\Rightarrow$ $\{y\} = F \subset X$ and $x \in X$, $x \notin F$

Since $(X,\tau)$ is $[R]$

$\Rightarrow$ $\exists$ disjoint open sets $G, H$ with $F \subset G \wedge x \in H$

Since $y \in F \Rightarrow y \in G$

$\Rightarrow$ $\exists$ disjoint open sets $G, H$ with $y \in G \wedge x \in H$

$\Rightarrow$ $(X,\tau)$ is $T_2$-space

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**Definition (4.3): (Normal Space)** **الفضاء السّوي**

We say that $(X,\tau)$ is **normal space** denoted by $[N]$ if $\forall$ disjoint closed sets $F_1, F_2$ in $X$, $\exists$ disjoint open sets $G, H$ with $F_1 \subset G \wedge F_2 \subset H$.

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**Example (4.3):** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Discuss whether $(X,\tau)$ is $[N]$ or not.

**Solution:** The closed sets are: $X, \{b, c\}, \{a\}, \emptyset$

$F_1 = \{b, c\}$ and $F_2 = \{a\}$, we have $G = \{b, c\}$ and $H = \{a\}$ are disjoint open set with $F_1 \subset G \wedge F_2 \subset H$

$\Rightarrow$ $(X,\tau)$ is $[N]$

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**Theorem (4.3):** The property $(X,\tau)$-$[N]$ is a topological property.

**Proof:** Let $f: (X,\tau) \rightarrow (X^*, \tau^*)$ be a homeo.

Let $(X,\tau)$ is $[N]$

Let $F_1^*, F_2^*$ disjoint closed subset for $X^*$

Since $f$ is onto

$\Rightarrow\ \exists\ F_1, F_2 \subset X$, s.t. $F_1^* = f(F_1)$ and $F_2^* = f(F_2)$

Since $f$ is continuous

$\Rightarrow\ F_1 = f^{-1}(F_1^*), F_2 = f^{-1}(F_2^*)$ are closed sets in $X$

Since $f$ is (1-1)

$\Rightarrow\ F_1 \cap F_2 = f^{-1}(F_1^*) \cap f^{-1}(F_2^*)$

$= f^{-1}(F_1^* \cap F_2^*) = f^{-1}(\emptyset) = \emptyset$

Since $(X, \tau)$ is $[N]$

$\exists$ disjoint open sets $G, H$ s.t. $F_1 \subset G \wedge F_2 \subset H$

Since $f$ is open

$\Rightarrow\ G^* = f(G)$ and $H^* = f(H)$ are open in $X^*$

$G^* \cap H^* = f(G) \cap f(H)$

$= f(G \cap H) = f(\emptyset) = \emptyset$

Now, $F_1 \subset G \Rightarrow f(F_1) \subset f(G) \Rightarrow F_1^* \subset G^*$

$F_2 \subset H \Rightarrow f(F_2) \subset f(H) \Rightarrow F_2^* \subset H^*$

$\Rightarrow\ \forall$ disjoint closed sets $F_1^*, F_2^*$ in $X^*$,

$\exists$ disjoint open sets $G^*, H^*$ with $F_1^* \subset G^* \wedge F_2^* \subset H^*$

$\Rightarrow\ (X^*, \tau^*)$ is $[N]$

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**Remark (4.1):** The property $(X, \tau)$-$[N]$ is not hereditary.

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**Definition (4.4): ($T_4$- Space)**

We say that $(X, \tau)$ is $\boldsymbol{T_4}$**-space** if $(X, \tau)$ is $T_1$-space and $[N]$.

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**Example (4.4):** Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$. Discuss whether $(X, \tau)$ is $T_4$-space or not.

**Solution:** The closed sets are: $X, \{b\}, \{a\}, \emptyset$

$F_1 = \{b\}$ and $F_2 = \{a\}$, we have $G = \{b\}$ and $H = \{a\}$ are disjoint open set with $F_1 \subset G \wedge F_2 \subset H$

$\Rightarrow$ $(X, \tau)$ is $[N]$

Also, $(X, \tau)$ is $T_1$-space, because For $a, b \in X$, $a \neq b$ and $\exists^{open}$ $G = \{a\}$, $H = \{b\}$; $a \in G, b \notin G \ \wedge \ a \notin H, b \in H$

$\Rightarrow$ $(X, \tau)$ is $T_4$-space

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**Theorem (4.4):** Every $T_4$-space is $T_3$-space.

**Proof:** Let $(X, \tau)$ be $T_4$-space

We need to show that $(X, \tau)$ is $T_3$-space

Let $F \subset X$ and $x \notin F$

Since $(X, \tau)$ is $T_1$-space

$\Rightarrow$ $F_1 = \{x\}$ is closed

Since $F, F_1 \subset X$ and $F \cap F_1 = \emptyset$

Thus, we have $F, F_1 \subset X$ are disjoint and $(X, \tau)$ is $[N]$

$\Rightarrow$ $\exists$ disjoint open sets $G, H$ with $F \subset G \wedge x \in H$

$\Rightarrow$ $(X, \tau)$ is $[R]$ and $(X, \tau)$ is $T_1$-space $\Rightarrow$ $(X, \tau)$ is $T_3$-space.

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**Theorem (4.5):** Every compact $T_2$-space is $T_4$-space.

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**Lemma (4.1): (Urysohn's Lemma)** قضية يوريسون

A topological space $(X, \tau)$ is $[N]$ iff $\forall$ disjoint closed sets $F_1, F_2$ in $X$, $\forall$ closed interval $[a, b] \subset R$, $\exists$ $f$ is continuous function as follows:

$f: X \to [a, b]$ s.t. $f(F_1) = \{a\}$, $f(F_2) = \{b\}$.

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**Definition (4.5): (Tietze's Axiom)** بديهية تايتز

$\forall$ separable sets $A, B$, $\exists$ two disjoint open sets $G, H$ such that $A \subset G \wedge B \subset H$.

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**Definition (4.6): (Completely Normal Spaces)** الفضاءات كاملة السّوية

We say that $(X,\tau)$ is **complete normal space** denoted by $[CN]$ if satisfies Tietze's axiom.

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**Example (4.5):** Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,\{a\},\{b,c\},X\}$. Discuss whether $(X,\tau)$ is $[CN]$-space or not.

**Solution:** Since $A=\{a\}$, $B=\{b,c\}$ are closed and open

$\Rightarrow$ $A,B$ are separable

$\exists$ two disjoint open sets $G=\{a\}$, $H=\{b,c\}$ such that $A\subset G, B\subset H$

$\Rightarrow$ $(X,\tau)$ is $[CN]$

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**Theorem (4.6):** The property $(X,\tau)$-$[CN]$ is a topological property.

**Proof:** Let $f:(X,\tau)\to(X^*,\tau^*)$ be a homeo.

Let $(X,\tau)$ is $[CN]$

Let $A^*,B^*$ are disjoint separable subset for $X^*$

Since $f$ is continuous and (1-1)

$\Rightarrow$ $\exists$ $A,B$ are disjoint separable subset for $X$ s.t. $A^*=f(A)$ and $B^*=f(B)$

Since $(X,\tau)$ is $[CN]$

$\Rightarrow$ $\exists$ two disjoint open sets $G,H$ such that $A\subset G\wedge B\subset H$

Since $f$ is open

$\Rightarrow$ $G^*=f(G)$ and $H^*=f(H)$ are open in $X^*$

$G^*\cap H^*=f(G)\cap f(H)$

$\qquad\qquad=f(G\cap H)=f(\emptyset)=\emptyset$

Since $f$ (1-1)

$A\subset G \;\Rightarrow\; f(A)\subset f(G) \;\Rightarrow\; A^*\subset G^*$

$B\subset H \;\Rightarrow\; f(B)\subset f(H) \;\Rightarrow\; B^*\subset H^*$

$\Rightarrow\forall$ disjoint separable subset $A^*,B^*$ for $X^*$

$\exists$ disjoint open sets $G^*,H^*$ with $A^*\subset G^*\wedge B^*\subset H^*$