$\Rightarrow$ $(X^*, \tau^*)$ is $[CN]$

$\Rightarrow$ $[CN]$ is a topological property.

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**Remark (4.2):** The property $(X, \tau)$-$[CN]$ is not hereditary.

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**Theorem (4.7):** The topological space $(X, \tau)$ is $[CN]$-space iff every subspace of $(X, \tau)$ is $[N]$-space.

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**Definition (4.7): ($T_5$ - Spaces)**

We say that $(X, \tau)$ is **$T_5$ - space** if is $T_1$-space and $[CN]$.

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**Definition (4.8): (Completely Regular Spaces)** الفضاءات كاملة الانتظام

We say that $(X, \tau)$ is **complete regular space** denoted by $[CR]$ if satisfies the following axiom:

$\forall^{closed} F \subset X, \forall x \in X, x \notin F, \exists$ continuous function $f: X \to [0,1]$ such that $f(F) = \{1\}, f(x) = 0$

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**Definition (4.9): ($T_{3\frac{1}{2}}$ – Spaces (Tychonoff Space) )** فضاء تيخونوف

We say that $(X, \tau)$ is **Tychonoff space** denoted by $T_{3\frac{1}{2}}$ -space if $(X, \tau)$ is $T_1$-space and $[CR]$.

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**Theorem (4.8):** Every metric space is Tychonoff space.

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**Theorem (4.9):**

**(1)** The property $[CR]$ is hereditary.

**(2)** The property $[CR]$ is a topological property.

**Proof:**

**(1)** Let $(X, \tau)$ be $[CR]$

We need to show that $(X^*, \tau^*)$ is $[CR]$

Let $x \in X^*$ and $F^* \subset X^*$ be closed such that $x \notin F^*$

Since $X^* \subset X$

$\Rightarrow$ $\exists$ closed $F \subset X$ such that $F^* = F \cap X^*$

Now, $x \in X^*, x \notin F^*$ $\Rightarrow$ $x \notin F$

Since $X$ is $[CR]$

$\exists$ continuous function $f: X \to [0,1]$ such that $f(F) = \{1\}$, $f(x) = 0$

$\Rightarrow$ $f^*: X^* \to [0,1]$ which defined as

$f^*(x) = f(x)$, $\forall\, x \in X^*$ is continuous and satisfies

$f^*(F^*) = \{1\}$, $f^*(x) = 0$

$\Rightarrow$ $(X^*, \tau^*)$ is $[CR]$

$\Rightarrow$ $[CR]$ is a hereditary property.

**(2)** Let $f: X \to X^*$ be a homeo.

Let $(X, \tau)$ be $[CR]$

We need to show that $(X^*, \tau^*)$ is $[CR]$

Let $x^* \in X^*$ and $F^* \subset X^*$ be closed such that $x^* \notin F^*$

Since $f$ is continuous

$\Rightarrow$ $f^{-1}(F^*) = F \subset X$ is closed

Since $f$ is onto

$\Rightarrow$ $\exists\, x \in X, f(x) = x^*$

Since $f^{-1}$ is (1-1), $x^* \notin F^*$

$\Rightarrow$ $f^{-1}(x^*) \notin f^{-1}(F^*)$ $\Rightarrow$ $x \notin F$

We have $F \subset X$ is closed and $x \notin F$ but $(X, \tau)$ is $[CR]$

$\Rightarrow$ $\exists$ continuous function $g: X \to [0,1]$ such that $g(F) = \{1\}$, $g(x) = 0$

We have

$X^* \xrightarrow{f^{-1}} X \xrightarrow{g} [0,1]$

$\Rightarrow\ g_\circ f^{-1}: X^* \to [0,1]$

$g_\circ f^{-1}$ is continuous (The composite function of two continuous functions is continuous)

Now, $(g_\circ f^{-1})(F^*) = g\big(f^{-1}(F^*)\big) = g(F) = \{1\}$

$\therefore\ (g_\circ f^{-1})(F^*) = \{1\}$

$(g_\circ f^{-1})(x^*) = g\big(f^{-1}(x^*)\big) = g(x) = 0$

$\therefore\ (g_\circ f^{-1})(x^*) = 0$

$\Rightarrow\ (X^*, \tau^*)$ is $[CR]$.

$\Rightarrow\ [CR]$ is a topological property.

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