We can take any counting number, square it, and end up with a nice neat number. But the process doesn't always work nicely when going backwards. For instance, consider $\sqrt{3}$, the square root of three. There is no nice neat number that squares to 3, so $\sqrt{3}$ cannot be simplified as a nice whole number. We can deal with $\sqrt{3}$ in either of two ways: If we are doing a word problem and are trying to find, say, the rate of speed, then we would grab our calculators and find the decimal approximation of $\sqrt{3}$:

$$\sqrt{3} \approx 1.732050808$$

Then we'd round the above value to an appropriate number of decimal places and use a real-world unit or label, like "1.7 ft/sec". On the other hand, we may be solving a plain old math exercise, something having no "practical" application. Then they would almost certainly want us to give the "exact" value, so we'd write our answer as being simply "$\sqrt{3}$".

Since most of what you'll be dealing with will be square roots (that is, second roots), most of this lesson will deal with them specifically.

Simplifying Square-Root Terms

To simplify a term containing a square root, we "take out" anything that is a "perfect square"; that is, we factor inside the radical symbol and then we take out in front of that symbol anything that has two copies of the same factor. For instance, 4 is the square of 2, so the square root of 4 contains two copies of the factor 2; thus, we can take a 2 out front, leaving nothing (but an understood 1) inside the radical, which we then drop:

$$\sqrt{4} = \sqrt{2^2}$$
$$= \sqrt{2 \times 2} = 2$$

Similarly, 49 is the square of 7, so it contains two copies of the factor 7:

$$\sqrt{49} = \sqrt{7^2}$$
$$= \sqrt{7 \times 7} = 7$$

And 225 is the square of 15, so it contains two copies of the factor 15, so:

$$\sqrt{225} = \sqrt{15^2}$$
$$= \sqrt{15 \times 15} = 15$$

Note that the value of the simplified radical is *positive*. While either of +2 and −2 might have been squared to get 4, "the square root of four" is *defined* to be *only* the positive option, +2. That is, the definition of the square root says that the square root will spit out *only* the positive root.

On a side note, let me emphasize that "evaluating" an expression (to find its one value) and "solving" an equation (to find its one or more, or no, solutions) are two very different things. In the first case, we're simplifying to find the one defined value for an expression. In the second case, we're looking for any and all values what will make the original equation true.

So, for instance, when we solve the equation $x^2 = 4$, we are trying to find *all* possible values that *might* have been squared to get 4. But when we are just simplifying the expression $\sqrt{4}$, the ONLY answer is "2"; this positive result is called the "principal" root. (Other roots, such as −2, can be defined using graduate-school topics like "complex analysis" and "branch functions", but you won't need that for years, if ever.)

Oftentimes the argument of a radical is not a perfect square, but it may "contain" a square amongst its factors. To simplify this sort of radical, we need to factor the argument (that is, factor whatever is inside the radical symbol) and "take out" one copy of anything that is a square. That is, we find anything of which we've got a pair inside the radical, and we move one copy of it out front. When doing this, it can be helpful to use the fact that we can switch between the multiplication of roots and the root of a multiplication. In other words, we can use the fact that radicals can be manipulated similarly to powers:

$$(ab)^n = a^n b^n$$

And

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

Example: *Simplify* $\sqrt{\mathbf{144}}$

There are various ways I can approach this simplification. One would be by factoring and then taking two different square roots. In particular, I'll start by factoring the argument, 144, into a product of squares:

144 = 9 × 16

Each of 9 and 16 is a square, so each of these can have its square root pulled out of the radical. The square root of 9 is 3 and the square root of 16 is 4. Then:

$$\sqrt{144} = \sqrt{9 \times 16}$$
$$= \sqrt{9}\sqrt{16}$$
$$= 3 \times 4 = 12$$

Then my solution is:

$$\sqrt{144} = 12$$

Another way to do the above simplification would be to remember our squares. You probably already knew that $12^2 = 144$, so obviously the square root of 144 must be 12. But my steps above show how you can switch back and forth between the different formats (multiplication inside one radical, versus multiplication of two radicals) to help in the simplification process.

In case you're wondering, products of radicals are customarily written as shown above, using "multiplication by juxtaposition", meaning "they're put right next to one another, which we're using to mean that they're multiplied against each other". You *could* put a "times" symbol between the two radicals, but this isn't standard.

- ***Simplify*** $\sqrt{24}\sqrt{6}$

Neither of 24 and 6 is a square, but what happens if I multiply them inside one radical?

$$\sqrt{24}\sqrt{6} = \sqrt{24 \times 6} = \sqrt{144}$$

Now I *do* have something with squares in it, so I can simplify as before:

$$\sqrt{144} = \sqrt{12 \times 12} = 12$$

- ***Simplify*** $\sqrt{75}$

The argument of this radical, 75, factors as:

$$75 = 3 \times 5 \times 5$$

This factorization gives me two copies of the factor 5, but only one copy of the factor 3. Since I have two copies of 5, I can take 5 out front. Since I have only the one copy of 3, it'll have to stay behind in the radical. Then my answer is:

$$\sqrt{75} = \sqrt{3 \times 25}$$

$$= \sqrt{3}(5) = 5\sqrt{3}$$

This answer is pronounced as "five, times root three", "five, times the square root of three", or, most commonly, just "five, root three".

- ***Simplify*** $\sqrt{72}$

Since 72 factors as 2×36, and since 36 is a perfect square, then:

$$\sqrt{72} = \sqrt{2 \times 36}$$

$$= \sqrt{2 \times 6 \times 6} = 6\sqrt{2}$$

Since there had been only one copy of the factor 2 in the factorization 2 × 6 × 6, the left-over 2 couldn't come out of the radical and had to be left behind.

- ***Simplify*** $\sqrt{\mathbf{4,500}}$

The argument, 4,500, factors as:

$$45 \times 100$$

$$= 5 \times 9 \times 100$$

I could continue factoring, but I know that 9 and 100 are squares, while 5 isn't, so I've gone as far as I need to. I'm ready to evaluate the square root:

$$\sqrt{4{,}500} = \sqrt{45 \times 100}$$

$$= \sqrt{5 \times 9 \times 100}$$

$$= \sqrt{5} \times 3 \times 10$$

$$= 30\sqrt{5}$$

Yes, I used "times" in my work above. No, you wouldn't include a "times" symbol in the final answer. I was using the "times" to help me keep things straight *in my work*. I used regular formatting for my hand-in answer. When doing your work, use whatever notation works well for you.

Conversion between radicals and fractional exponents

Radicals and fractional exponents are alternate ways of expressing the same thing. In the table below we show equivalent ways to express radicals: with a root, with a rational exponent, and as a principal root.

Radical Form	Exponent Form	Principal Root
$\sqrt{16}$	$16^{\frac{1}{2}}$	4
$\sqrt{25}$	$25^{\frac{1}{2}}$	5
$\sqrt{100}$	$100^{\frac{1}{2}}$	10

Let's look at some more examples, but this time with cube roots. Remember, cubing a number raises it to the power of three. Notice that in the examples in the table below, the denominator of the rational exponent is the number 3.

Radical Form	Exponent Form	Principal Root
$\sqrt[3]{8}$	$8^{\frac{1}{3}}$	2
$\sqrt[3]{125}$	$125^{\frac{1}{3}}$	5
$\sqrt[3]{1000}$	$1000^{\frac{1}{3}}$	10

These examples help us model a relationship between radicals and rational exponents: namely, that the n^{th} root of a number can be written as either $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$

Radical Form	Exponent Form
$\sqrt{x}$	$x^{\frac{1}{2}}$
$\sqrt[3]{x}$	$x^{\frac{1}{3}}$
$\sqrt[4]{x}$	$x^{\frac{1}{4}}$
…	…
$\sqrt[n]{x}$	$x^{\frac{1}{n}}$

In the table above, notice how the denominator of the rational exponent determines the index of the root. So, an exponent of $\frac{1}{2}$ translates to the square root, an exponent of $\frac{1}{5}$ translates to the fifth root or $\sqrt[5]{.}$, and $\frac{1}{8}$ translates to the eighth root or $\sqrt[8]{.}$.

Example: express $(2x)^{\frac{1}{3}}$ in radical form

Solution: Rewrite the expression with the fractional exponent as a radical. The denominator of the fraction determines the root, in this case the cube root.

$$\sqrt[3]{2x}$$

The parentheses in $(2x)^{\frac{1}{3}}$ indicate that the exponent refers to everything within the parentheses.

$$(2x)^{\frac{1}{3}} = \sqrt[3]{2x}$$

Example: express $2x^{\frac{1}{3}}$ in radical form

Solution: Rewrite the expression with the fractional exponent as a radical. The denominator of the fraction determines the root, in this case the cube root.

$$2\sqrt[3]{x}$$

The exponent refers only to the part of the expression immediately to the left of the exponent, in this case x, but not the 2.

$$2x^{\frac{1}{3}} = 2\sqrt[3]{x}$$

H.W.:

1- Write $\sqrt[4]{81}$as an expression with a rational exponent.

2- Express $4\sqrt[3]{xy}$ with rational exponents.

Example: Rewrite the radicals using a rational exponent, then simplify your result.

1- $\sqrt[3]{a^6}$

$$\sqrt[3]{a^6} = a^{\frac{6}{3}}$$

simplify the exponent

$$a^{\frac{6}{3}} = a^2$$

2- $\sqrt[12]{16^3}$

$$\sqrt[12]{16^3} = 16^{\frac{3}{12}} = 16^{\frac{1}{4}}$$

Simplify the expression using rules for exponents.

$$16 = 2^4$$

$$16^{\frac{1}{4}} = 2^{4\frac{1}{4}}$$

$$= 2^1 = 2$$

$$\sqrt[12]{16^3} = 2$$

H.W.: Rewrite the expressions using a radical.

1. $x^{\frac{2}{3}}$

2. $5^{\frac{4}{7}}$

Example: simplify $\sqrt{63}$

$$\sqrt{63} = \sqrt{7 \times 9}$$
$$\sqrt{7 \times 3^2}$$

$$\sqrt{7} \times \sqrt{3^2}$$

$$3\sqrt{7}$$

Example: simplify $\sqrt{9x^6y^4}$

Solution:

$$\sqrt{9x^6y^4} = \sqrt{3.3.x^3.x^3.y^2.y^2}$$

$$= \sqrt{x^2.(x^3)^2.(y^2)^2}$$

$$= \sqrt{3^2}.\sqrt{(x^3)^2}.\sqrt{(y^2)^2}$$
$$= 3x^3y^2$$

Example: simplify $\sqrt{x^2 - 6x + 9}$

Solution: first we factor the radicand:

$$x^2 - 6x + 9 = (x - 3)^2$$

Then, we rewrite the radical expression and take the square root:

$$\sqrt{x^2 - 6x + 9} = \sqrt{(x - 3)^2} = |x - 3|$$

Example: simplify $(36x^4)^{\frac{1}{2}}$

Solution: Rewrite the expression with the fractional exponent as a radical.

$$\sqrt{36x^4}$$

Find the square root of both the coefficient and the variable.

$$\sqrt{6^2x^4} = \sqrt{6^2}.\sqrt{x^4}$$
$$= \sqrt{6^2}.\sqrt{(x^2)^2}$$
$$= 6.x^2$$

Example : simplify $\sqrt[3]{-24a^5}$

Solution: Factor −24 to find perfect cubes. Here, −1 and 8 are the perfect cubes.

$$\sqrt[3]{-1.8.3.a^5}$$
$$\sqrt[3]{(-1)^3.2^3.3.a^3.a^2}$$
$$\sqrt[3]{(-1)^3}.\sqrt[3]{2^3}.\sqrt[3]{a^3}.\sqrt[3]{3.a^2}$$
$$= -2a\sqrt[3]{3a^2}$$

H.W.: simplify

1. $\sqrt{49x^{10}y^8}$

2. $\sqrt[4]{81x^8y^3}$

Solving radical equations:

Example1: solve $\sqrt{x-2} + 1 = 6$

Solution: isolate radical so it's the only term on 1 side of the equation

$$\sqrt{x-2} = 5$$

Square both sides of the equation

$$\left(\sqrt{x-2}\right)^2 = 5^2$$

Simplify and solve

$$x - 2 = 25$$
$$x = 27$$

Check solution to make sure it's not an extraneous root

Example 2: solve $\sqrt{2x-1} + 7 = 4$

Solution: $\sqrt{2x-1} = -3$

$$\left(\sqrt{2x-1}\right)^2 = (-3)^2$$

$$2x - 1 = 9$$
$$x = 5$$

Example3: Solve the radical Equation Below.

$$\sqrt{x-1} + 2 = 5$$

Solution: $\sqrt{x-1} = 3$

$$\left(\sqrt{x-1}\right)^2 = 3^2$$

$$x - 1 = 9$$
$$x = 10$$

Example4: Solve the radical Equation Below

$$2\sqrt{3x+2}+1=11$$

Solution:

$$\sqrt{3x+2}=5$$

$$\left(\sqrt{3x+2}\right)^2=5^2$$

$$3x+2=25$$

$$3x=23$$

$$x=\frac{23}{3}$$

Example5: Solve the following radical equation:

$$\sqrt{3x-11}=3-x$$

Solution:

$$\left(\sqrt{3x-11}\right)^2=(3-x)^2$$

$$3x-11=x^2-6x+9$$

$$3x=x^2-6x+20$$

$$0=x^2-9x+20$$

$$(x-4)(x-5)=0$$

$$x=4, x=5$$

Solving radical equations:

Example1: solve $\sqrt{x-2}+1=6$

Solution: isolate radical so it's the only term on 1 side of the equation

$$\sqrt{x-2}=5$$

Square both sides of the equation

$$\left(\sqrt{x-2}\right)^2=5^2$$

Simplify and solve

$$x-2=25$$
$$x=27$$

Check solution to make sure it's not an extraneous root

Example 2: solve $\sqrt{2x-1}+7=4$

Solution: $\sqrt{2x-1}=-3$

$$\left(\sqrt{2x-1}\right)^2=(-3)^2$$

$$2x-1=9$$
$$x=5$$

Example3: Solve the radical Equation Below.

$$\sqrt{x-1}+2=5$$

Solution: $\sqrt{x-1}=3$

$$\left(\sqrt{x-1}\right)^2=3^2$$

$$x-1=9$$

$$x = 10$$

Example4: Solve the radical Equation Below

$$2\sqrt{3x+2} + 1 = 11$$

Solution:

$$\sqrt{3x+2} = 5$$

$$\left(\sqrt{3x+2}\right)^2 = 5^2$$

$$3x + 2 = 25$$

$$3x = 23$$

$$x = \frac{23}{3}$$

Example5: Solve the following radical equation:

$$\sqrt{3x-11} = 3 - x$$

Solution:

$$\left(\sqrt{3x-11}\right)^2 = (3-x)^2$$
$$3x - 11 = x^2 - 6x + 9$$
$$3x = x^2 - 6x + 20$$

$$0 = x^2 - 9x + 20$$
$$(x-4)(x-5) = 0$$
$$x = 4, x = 5$$