Solution:

(1) Let $G^* \ni f(x) = a$ be any point

$\forall^{open}\ G \ni x, f(G) = \{a\} \ni a$ and $f(G) \subset \{a\}$

$\Rightarrow\ \exists^{open}\ G \ni x, such\ that\ f(G) \subset \{a\}$

$\Rightarrow\ f$ is continuous

(2) f is open, since $f(x) = \{a\}$, $\forall\ E \subset X$

$\Rightarrow\ f(G) = \{a\}$ is open, $\forall^{open}\ G \subset X$

$\Rightarrow\ f$ is open

(3) We have $f(x_1) = f(x_2) = a$ while $x_1 \neq x_2$

$f(x_1) = f(x_2)\ \nRightarrow\ x_1 = x_2$

$\Rightarrow\ f$ is not (1-1)

(4) Also f is not onto

$\Rightarrow\ f$ is not homeo.

صفة تبولوجية

Definition (1.3): We say that a property P is a **topological property** if P carried by a topological homeomorphism.

Theorem (1.3): The property perfect set is a topological property.

Proof: Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homo.

Let $E \subset X$ be a perfect set

We need to show $f(E) \subset X^*$ is perfect

Since E is perfect

$\Rightarrow\ E$ is dense in itself and closed

Since E dense in itself and f is continuous and (1-1)

$\Rightarrow\ f(E)$ is dense in itself ....(1)

Now, since E is closed

$\Rightarrow\ E^c$ is open

Since f is (1-1) & onto

$\Rightarrow\ f(E^c) = \left(f(E)\right)^c$

Since f is open

$\Rightarrow\ f(E^c)$ is open

$\Rightarrow\ \left(f(E)\right)^c$ is open

$\Rightarrow\ f(E)$ is closed …….(2)

From (1) and (2), we get

$f(E)$ is dense in itself and closed

$\Rightarrow\ f(E)$ is perfect.

$\therefore$ The perfect set is a topological property.

Theorem (1.4): The property locally compact is a topological property.

Proof: Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homo.

Let $E \subset X$ be locally compact

We need to show $f(E) \subset X^*$ is locally compact

Let $x^* \in f(E)$ be any point

Since f is onto

$\Rightarrow\ \exists\ x \in E; x^* = f(x)$

Since E is locally compact and $x \in E$

$\Rightarrow\ \exists$ a compact nbhd $N(x)$, since f is continuous and $N(x)$ is compact

$\Rightarrow\ f(N(x))$ is compact and $x^* \in f(N(x))$, therefore $\exists$ a compact nbhd $f(N(x))$ of x^*, $\forall\ x^* \in f(E)$

Hence $f(E)$ is locally compact.

$\therefore$ The locally compact is a topological property.

Definition (1.4): We say that the set E is an **isolated** (منعزلة) set iff $E \cap d(E) = \emptyset$.

Example (1.3): Let (R, d) be the usual metric space, and the set

$E = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \dots\}$. Is E isolated set?

Solution:

We have $d(E) = \{0\}$ and $E \cap d(E) = \emptyset$.

$\Rightarrow$ E is isolated set.

Example (1.4): Let $X = \{a, b, c, d, e\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}, X\}$ which of the following subsets of X is isolated $E_1 = \{b, d\}$, $E_2 = \{a, b\}$

Solution: We have

$d(E_1) = d(\{b, d\}) = \{e\}$ and $E_1 \cap d(E_1) = \emptyset$.

$\Rightarrow$ E_1 is isolated set.

We have $d(E_2) = d(\{a, b\}) = \{c, d, e\}$ and $E_2 \cap d(E_2) = \emptyset$.

$\Rightarrow$ E_2 is isolated set.

Theorem (1.5): The property isolated set is a topological property.

Proof: Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homo.

Let $E \subset X$ be an isolated set

We need to show $f(E) \subset X^*$ is isolated

Since f is onto

$\Rightarrow$ $\exists\, x \in E$ such that $x^* = f(x)$

Since E is isolated and $x \in E$

$\Rightarrow$ $x \notin d(E)$

$\Rightarrow$ $\exists^{open}\, G \ni x; (G \cap E) \backslash \{x\} = \emptyset$

Since f is open and (1-1)

$\Rightarrow$ $\exists^{open}\, f(G) \ni x^*; f[(G \cap E) \backslash \{x\}] = f(\emptyset)$

$\Rightarrow$ $\exists^{open}\, f(G) \ni x^*; f(G) \cap f(E) \backslash \{f(x)\} = \emptyset$

$\Rightarrow$ $\exists^{open}\, f(G) \ni x^*; f(G) \cap f(E) \backslash \{x^*\} = \emptyset$

$\Rightarrow$ $x^* \notin d(f(E)), \forall\, x^* \in f(E)$

$\Rightarrow$ $f(E) \cap d(f(E)) = \emptyset$