$\Rightarrow$ $f(E)$ is isolated.

$\therefore$ The isolated set is a topological property.

Theorem (1.6): The property countablly compact is a topological property.

Proof: Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homo.

Let $E \subset X$ be a countablly compact set

We need to show $f(E) \subset X^*$ is countablly compact

Let $A^* \subset f(E)$ be an infinite set

Since f is onto and (1-1)

$\Rightarrow$ $\exists^{infinite} A \subset E$ such that $A^* = f(A)$

Now E is countablly compact and $A \subset E$ is infinite set

$\Rightarrow$ $\exists\ x \in E$ and $x \in d(A)$

$\Rightarrow$ $\exists\ x \in E$ and $\forall^{open} G \ni x; (G \cap A) \setminus \{x\} \neq \emptyset$

Since f is (1-1) and open

$\Rightarrow$ $\exists\ f(x) \in f(E)$ and $\forall^{open} f(G) \ni f(x); f[(G \cap A) \setminus \{x\}] \neq f(\emptyset)$

$\Rightarrow$ $\exists\ x^* \in f(E)$ and $\forall^{open} G^* \ni x^*; f(G) \cap f(A) \setminus \{f(x)\} \neq \emptyset$

$\Rightarrow$ $\exists\ x^* \in f(E)$ and $\forall^{open} G^* \ni x^*; G^* \cap f(A) \setminus \{x^*\} \neq \emptyset$

$\Rightarrow$ $\exists\ x^* \in f(E)$ and $x^* \in d\big(f(A)\big) = d(A^*)$

$\Rightarrow$ $f(E)$ is countablly compact.

$\therefore$ The countablly compact is a topological property.

Example (1.5): Prove that the property "x is an interior point" is a topological property.

Solution: Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homo.

Let $E \subset X$ and let $x \in E^\circ$

We need to show $x^* = f(x) \in (f(E))^\circ$

Since f is open, we have

$f(E^\circ) \subseteq (f(E))^\circ$

Since $x \in E^{\circ}$

$\Rightarrow\ x^* = f(x) \in f(E^{\circ})$

$\Rightarrow\ x^* = f(x) \in ((f(E))^{\circ}$

Example (1.6): Prove that the property "x is a boundary point" is a topological property.

Solution: Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homo.

Let $E \subset X$ and let $x \in b(E) = \bar{E} - E^{\circ}$

$\Rightarrow\ x \in \bar{E}$ and $x \notin E^{\circ}$

We need to show $x^* = f(x) \in b\big(f(E)\big)$

Since $x \in \bar{E} - E^{\circ}$

$\Rightarrow\ f(x) \in f[\bar{E} - E^{\circ}]$

$\Rightarrow\ x^* \in f(\bar{E}) - f(E^{\circ})$

Since f is continuous and open

$\Rightarrow\ x^* \in \overline{f(E)} - (f(E))^{\circ}$

$\Rightarrow\ x^* = f(x) \in b\big(f(E)\big)$

Definition (1.5): We say that the property P is not topological property if P cannot be carried by a homeomorphism.

Example (1.7): Is the length property a topological property?

Solution: The length property is not topological property because the function $f: (R, d) \to (R, d)$ such that $f(x) = \frac{x}{4}$, $\forall\, x \in R$ where (R, d) the usual metric space is a homo. But $d(E) \neq d(f(E))$ because if $E = (a, b)$

$d(E) = d(a, b) = |b - a|$

$d\big(f(E)\big) = d(f(a, b))$

$$= d\left(\frac{a}{4}, \frac{b}{4}\right) = \left|\frac{b}{4} - \frac{a}{4}\right| = \frac{1}{4}|b - a| \neq |b - a| = d(E)$$

∴ The length property is not a topological property.

Example (1.8): Is the property Cauchy sequence a topological property?

Solution: The property Cauchy sequence is not topological property

Let $f: (R^*, d) \to (R^*, d)$ such that $f(x) = \frac{1}{x}$, $\forall\ x \neq 0$ is a homo.

Let $\{x_n\} = \left\{\frac{1}{n}\right\}$ be a sequence in (R^*, d) (domain)

We have $\left\{\frac{1}{n}\right\} \to 0 \ \Rightarrow\ \left\{\frac{1}{n}\right\}$ is Cauchy

But

$f(\{x_n\}) = \{f(x_n)\} = \left\{f\left(\frac{1}{n}\right)\right\} = \{n\}$ is not Cauchy in (R^*, d) (range)

∴ The Cauchy sequence is not a topological property.

Exercise (1.2): (Homework)

(1) Prove that there exists a continuous function $f: (R, \tau) \to (R, \tau)$ which is closed but not open.
