Chapter Two

الصفات الوراثية والصفات غير الوراثية

Hereditary and Non-Hereditary Properties

Definition (2.1): (Absolute Property) **الصفة المطلقة**

We say that the property P is an **absolute property** if P does not depend on the topological space which contains the set E with P.

Example (2.1): Is "E connected" an absolute property ?

Solution:

Let (X^*, τ^*) be a topological subspace of (X, τ)

Let $E \subset X$

We need to show that $(E \subset X)$ connected $\Leftrightarrow$ $(E \subset X^*)$ connected

Let $A, B \neq \emptyset$ disjoint subset of $E = A \cup B$

Now, consider

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) = (A \cap X^* \cap \bar{B}) \cup (\bar{A} \cap X^* \cap B)$$

$$= (A \cap \bar{B}^*) \cup (\bar{A}^* \cap B)$$

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) = (A^* \cap \bar{B}^*) \cup (\bar{A}^* \cap B^*)$$

Now, if

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) \neq \emptyset \quad \Leftrightarrow \quad (A^* \cap \bar{B}^*) \cup (\bar{A}^* \cap B^*) \neq \emptyset$$

That is E is connected in X $\leftrightarrow$ E connected in X^*

Definition (2.2): (Hereditary Property) **الصفة الوراثية**

We say that the property P is **hereditary property** if P satisfied in any topological subspace of the topological space (X, τ).

Example (2.2): Disprove that the property "E compact" is not hereditary property ?

Solution:

Let $(X,\tau) = (R,\tau)$ the usual topological

Let $E = [0,1]$ and $\tau^* = \{(a,b) \in [0,1], \emptyset, E\}$

We have E is a compact

But $I = (0,1) \subset E$ is not compact

Thus the compactness is not hereditary.

Example (2.3): The property "E open (E closed)" is not hereditary property.

Solution:

$X = \{a,b,c,d,e\}$ and

$\tau = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}, \{a,b,c,d\}, X\}$

$X^* = \{a,b,d\}$ and $\tau^* = \{\emptyset, \{a\}, \{a,d\}, X^*\}$

Now, we have

$\{a,b\}$ is open in X

But $\{a,b\}$ is not open in X^*

Also, $\{b\}$ is closed in X^*

But $\{b\}$ is not closed in X.

Example (2.4): Prove the property "(X,τ) is discrete" is a hereditary property.

Solution:

Let (X,τ) be a discrete topological space

Let (X^*,τ^*) be a topological subspace of (X,τ)

We need to show that (X^*,τ^*) is discrete topological space

Since $X^* \subset X$

$\Rightarrow\ \tau^* = \{G^* = G \cap X^* : G \in \tau\}$

If $E \in \tau, E \subset X^* \ \Rightarrow\ E \cap X^* = E$

$\forall^{open}\ G \in \tau, G \subset X^* \Rightarrow G \cap X^* = G \subset X^*$

$\Rightarrow \tau^* = \{G^* : G^* \subset X^*\}$

$\Rightarrow (X^*, \tau^*)$ discrete topological space.

Example (2.5): Prove the property "(X, τ) is connected" is not a hereditary property.

Solution:

Let (R, τ) be an usual topological space

Let (X^*, τ^*) be a topological subspace of (X, τ), where $X^* = R\backslash\{0\}$

It is clear that $R\backslash\{0\}$ is not connected subset of the connected space R.
