Chapter Three
بديهيات الفصل
The Separation Axioms

Definition (3.1): (T_0 – Space (Kolmogorov Space)) فضاء كولموغوروف

We say that (X, τ) is $\boldsymbol{T_0}$**-space** iff $\forall\, x, y \in X\ (x \neq y), \exists^{open}\, G \in \tau$ containing one of the points and not the other.

Example (3.1): Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$

Then (X, τ) is T_0-space. For $a, b \in X$, $a \neq b$ and $G = \{a\} \ni a$, $\{a\} \not\ni b$.

Example (3.2): The discrete topological space (X, τ) is T_0-space.

For $\forall\, x, y \in X, x \neq y$ we have $\{x\}$ is open and contains x but not y

Also, $\{y\}$ is open and contains y but not x

Example (3.3): Let $X = \{a, b\}$ and $\tau = \{\emptyset, X\}$ is not T_0-space

For $a, b \in X$, $a \neq b$, $\nexists^{open}\, G \in \tau$ containing one of the point but not the other.

Example (3.4): Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$

The space (X, τ) is not T_0-space for $b \neq c$ and $\forall^{open} G \ni b, G \ni c$ also

$\Rightarrow$ (X, τ) is not T_0 $-$space.

Theorem (3.1):

(1) The property "(X, τ) is T_0-space" is hereditary.

(2) The property "(X, τ) is T_0-space" is a topological property.

Proof:

(1) Let (X, τ) be a topological space

Let (X^*, τ^*) be a topological subspace of (X, τ)

We need to show that (X^*, τ^*) is T_0-space

Let $x, y \in X^*, x \neq y$

$\Rightarrow\ x, y \in X, x \neq y$

Since (X, τ) is T_0-space

$\Rightarrow\ \exists^{open}\ G \in \tau$, say $x \in G,\ y \notin G$

We have $\tau^* = \{G^* = G \bigcap X^* : G \in \tau\}$

$x \in G, x \in X^*\ \Rightarrow\ x \in G \bigcap X^*\ \Rightarrow\ x \in G^* \in \tau^*$

$y \notin G, y \in X^*\ \Rightarrow\ y \notin G \bigcap X^*\ \Rightarrow\ y \notin G^* \in \tau^*$

$\Rightarrow\ \exists^{open}\ G^* \in \tau^*; x \in G^*, y \notin G^*\ \ \forall\, x, y \in X^*\ (x \neq y)$

$\Rightarrow\ (X^*, \tau^*)$ is T_0-space.

$\Rightarrow\ T_0$-space is a hereditary property.

(2) Let $f: X \to X^*$ be a homeo.

Let (X, τ) be T_0-space

We need to show that (X^*, τ^*) is T_0-space

Let $x^*, y^* \in X^*, x^* \neq y^*$

Since f is onto

$\exists\ x, y \in X, x^* = f(x) \wedge\ y^* = f(y)$

Since f is (1-1) and $f(x) \neq f(y)$

$\Rightarrow\ x \neq y$

$\Rightarrow\ x, y \in X\ (x \neq y)$ and (X, τ) is T_0-space

$\Rightarrow\ \exists^{open}\ G \in \tau\ \ (say)\ x \in G, y \notin G$

Since f is open

$\Rightarrow\ f(G) = G^*$ is open in (X^*, τ^*)

Now, $x \in G\ \ \Rightarrow\ \ f(x) \in f(G)\ \ \Rightarrow\ \ x^* \in G^*$

$y \notin G\ \ \Rightarrow\ \ f(y) \notin f(G)\ \ \Rightarrow\ \ y^* \notin G^*$

$\Rightarrow\ \exists^{open}\ G^* \in \tau^*;\ \ x^* \in G^*, y^* \notin G^*$

$\Rightarrow\ (X^*, \tau^*)$ is T_0-space.

$\Rightarrow\ T_0$-space is a topological property.

Theorem (3.2):

The topological space (X,τ) is T_0-space iff $\forall\, x,y \in X$ $(x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}})$.

Proof:

Let $\forall\, x,y \in X$ $x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$

Since $\overline{\{x\}} \neq \overline{\{y\}}$

$\Rightarrow \exists\, z \in X$ belong to one of them and not belongs the other

Assume $z \in \overline{\{x\}}$, $z \notin \overline{\{y\}}$

If $x \in \overline{\{y\}}$

$\Rightarrow \{x\} \subseteq \overline{\{y\}}$

$\Rightarrow \overline{\{x\}} \subseteq \overline{\overline{\{y\}}} = \overline{\{y\}}$

$\Rightarrow z \in \overline{\{y\}}$ C!

So $x \notin \overline{\{y\}}$

$\Rightarrow x \in \overline{\{y\}}^c$

Since $\overline{\{y\}}$ is closed then $\overline{\{y\}}^c$ is open.

Thus we obtain an open set contains x and does not contains y

$\Rightarrow (X,\tau)$ is T_0-space

Conversely,

Let (X,τ) is T_0-space,

We have $\forall\, x,y \in X$ $(x \neq y), \exists^{open}\, G \in \tau$, $x \in G$ and $y \notin G$

Thus G^c is closed, $y \in G^c$ and $x \notin G^c$

By definition of $\overline{\{y\}}$ (the intersection of all closed subsets contains $\{y\}$)

Thus $y \in \overline{\{y\}}$

But $x \notin \overline{\{y\}}$ since $x \notin G^c$

$\therefore\ \overline{\{x\}} \neq \overline{\{y\}}$

Definition (3.2): ($\boldsymbol{T_1}$ – Space (Fréchet space)) **فضاء فريشيه**

We say that (X,τ) is $\boldsymbol{T_1}$**-space** iff $\forall\, x,y \in X$ $(x \neq y), \exists^{open}\, G,H \in \tau$ such that $x \in G, y \notin G \ \wedge\ y \in H, x \notin H$.