Example (3.5): Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, X\}$

Then (X, τ) is T_1-space. For $a, b \in X$, $a \neq b$ and $\exists^{open}\, G = \{a\}$, $H = \{b\}$; $a \in G, b \notin G \;\wedge\; a \notin H, b \in H$

Example (3.6): Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$

(X, τ) is not T_1-space. For $a, b \in X$, $a \neq b$ and $\nexists^{open}\, G$, s.t. $b \in G, a \notin G$

Example (3.7): The usual topological space (R, τ) is T_1-space

$\forall\, a, b \in R, a \neq b, \exists$ always open sets $G, H, s.t.\; a \in G, b \notin G \;\wedge\; b \in H, a \notin H$

Corollary (3.1): $(X, \tau) - T_1 \;\Rightarrow\; (X, \tau) - T_0$

Proof: Let (X, τ) be T_1-space

$\Rightarrow\; \forall\, x, y \in X\; (x \neq y), \exists^{open}\, G, H, s.t.\; x \in G, y \notin G \;\wedge\; y \in H, x \notin H$

$\Rightarrow\; \forall\, x, y \in X\; (x \neq y), \exists^{open}\, G, x \in G, y \notin G \;\wedge\; \exists^{open}\, H, y \in H, x \notin H$

$\Rightarrow\; \forall\, x, y \in X\; (x \neq y), \exists^{open}\, G, x \in G, y \notin G \;\wedge$

$\forall\, x, y \in X\; (x \neq y), \exists^{open}\, H, y \in H, x \notin H$

$\Rightarrow\; (X, \tau) - T_0 \;\wedge\; (X, \tau) - T_0$

$\Rightarrow\; (X, \tau) - T_0$

Theorem (3.3):

(1) The property "$(X, \tau) - T_1$" is hereditary.

(2) The property "$(X, \tau) - T_1$" is a topological property.

Proof.

(1) Let (X, τ) be a topological space

Let (X^*, τ^*) be a topological subspace of (X, τ)

We need to show that (X^*, τ^*) is T_1-space

Let $x, y \in X^*$, $(x \neq y)$

$\Rightarrow\; x, y \in X$, $(x \neq y)$ (since $X^* \subset X$)

Since (X, τ) is T_1-space

$\Rightarrow\ \exists\, G, H \in \tau$ such that $x \in G, y \notin G\ \wedge\ x \notin H, y \in H$

We have

$\tau^* = \{G^* = G \cap X^* : G \in \tau\}$

$\tau^* = \{H^* = H \cap X^* : H \in \tau\}$

$x \in G, x \in X^*\ \Rightarrow\ x \in G \cap X^*\ \Rightarrow\ x \in G^*$

$y \notin G, y \in X^*\ \Rightarrow\ y \notin G \cap X^*\ \Rightarrow\ y \notin G^*$

$x \notin H, x \in X^*\ \Rightarrow\ x \notin H \cap X^*\ \Rightarrow\ x \notin H^*$

$y \in H, y \in X^*\ \Rightarrow\ y \in H \cap X^*\ \Rightarrow\ y \in H^*$

$\Rightarrow\ \exists\, G^*, H^* \in \tau^*; x \in G^*, y \notin G^*\ \wedge\ x \notin H^*, y \in H^*$

$\forall\, x, y \in X^*\ (x \neq y)$

$\Rightarrow\ (X^*, \tau^*)$ is T_1-space.

$\Rightarrow\ T_1$-space is a hereditary property.

(2) Let $f : (X, \tau) \to (X^*, \tau^*)$ be a homeo.

Let (X, τ) be T_1-space

Let $x^*, y^* \in X^*, x^* \neq y^*$

Since f is onto

$\exists\, x, y \in X, x^* = f(x), y^* = f(y)$

Since f is (1-1) and $x^* \neq y^*$

$\Rightarrow\ x \neq y$

$\Rightarrow\ x, y \in X, x \neq y$ and (X, τ) is T_1-space

$\Rightarrow\ \exists\, G, H \in \tau$ s.t. $x \in G, y \notin G\ \wedge\ x \notin H, y \in H$

Since f is open

$\Rightarrow f(G) = G^*, f(H) = H^*$ is open in (X^*, τ^*)

Now, $x \in G\ \Rightarrow\ f(x) \in f(G)\ \Rightarrow\ x^* \in G^*$

$y \notin G\ \Rightarrow\ f(y) \notin f(G)\ \Rightarrow\ y^* \notin G^*$

$x \notin H\ \Rightarrow\ f(x) \notin f(H)\ \Rightarrow\ x^* \notin H^*$

$y \in H\ \Rightarrow\ f(y) \in f(H)\ \Rightarrow\ y^* \in H^*$

$\Rightarrow \ \exists\, G^*, H^* \in \tau^*; x^* \in G^*, y^* \notin G^* \ \wedge \ x^* \notin H^*, y^* \in H^*$

$\Rightarrow$ (X^*, τ^*) is T_1-space.

$\Rightarrow$ T_1-space is a topological property.

Theorem (3.4): The topological space (X, τ) is T_1-space iff $\{x\}$ closed $\forall\, x \in X$.

Proof:

Let (X, τ) be T_1-space. Let $p \in X$. We have to show $\{p\}$ is closed.

Let $x \in \{p\}^c \ \Rightarrow \ x \neq p$

Since (X, τ) is T_1-space then $\exists^{open}\ G_x, p \notin G_x$ and $x \in G_x$

Thus $x \in G_x \subseteq \{p\}^c$

$\therefore\ \{p\}^c = \bigcup\{G_x : x \in \{p\}^c\}$

Thus $\{p\}^c$ is open $\Rightarrow$ $\{p\}$ is closed

Conversely:

Let $\{p\}$ is closed $\forall\, p \in X$

Assume $x, y \in X, x \neq y$

Now, $x \neq y \ \Rightarrow \ x \in \{y\}^c \ \Rightarrow \ \{y\}^c$ is open and contains x but not contains y

Similarly, $\{x\}^c$ is open , contains y and $x \notin \{x\}^c$

$\therefore$ (X, τ) is T_1-space

Example (3.8): Let $X = N = \{1,2, \dots, n, n+1, \dots\}$ and $\tau = \{\emptyset, X, \{1,2, \dots . n\}\}$

Then (X, τ) is not T_1-space. For

Case (1):

$1,2 \in X$, $(1 \neq 2)$ but every open $G \ni 2$ contains also 1

i.e. $\forall^{open}\ G \ni 2, G \ni 1$

$\nexists^{open}\ G \ni 2, G \not\ni 1$

$\Rightarrow$ (X, τ) is not T_1-space

Case (2):

According to above theorem, we have

$\forall\, n \in X$, $\{n\}$ is not closed

$\Rightarrow$ (X, τ) is not T_1-space

While (X, τ) is T_0-space for

If $m, n \in X$ $(m \neq n)$ put $m < n$

$\exists^{open}\, G = \{1,2, \dots, m\} \ni m,\, G \not\ni n$

$\Rightarrow$ $\forall\, m, n \in X, (m \neq n), \exists^{open}\, G$ containing one of the points only.

Remark (3.1): From the above example we conclude that $T_0 \not\Rightarrow T_1$.

Example (3.9): Let $X = R$, $\tau = \{\emptyset, E \subseteq R \colon E^c\ is\ finite\}$.

Proof:

Let $p \in R$ by any point

$\Rightarrow$ $\{p\} \subset X$ is finite

$\Rightarrow$ $\{p\}^c \subset X$ is open

$\Rightarrow$ $\{p\}$ closed

$\Rightarrow$ (R, τ) is T_1-space

Corollary (3.2): Every finite T_1-space is discrete.

Proof:

Let (X, τ) be finite T_1-space

We need to show that (X, τ) is discrete

Let $A_i \subset X$

$\Rightarrow$ A_i is finite

$\Rightarrow$ $A_i = \bigcup_{i=1}^{n} \{a_i\}$

Since (X, τ) is T_1-space

$\{a_i\}$ is closed $\forall\ i = 1,2, \dots, n$

$\Rightarrow$ $A_i = \bigcup_{i=1}^{n} \{a_i\}$ is closed

$\Rightarrow$ $\forall\, A_i \subset X,\ \ A_i$ closed , $\ i = 1,2, \dots, n$