Scalars:

Scalar quantities or simply **scalars** are physical quantities that can be described by a single pure number (a *scalar*, typically a real number), accompanied by a unit of measurement, as in "10 cm" (ten centimeters).[1] Examples of scalar are length, mass, charge, volume, and time. Scalars may represent the magnitude of physical quantities, such as speed is to velocity. Scalars do not represent a direction.

Scalars are unaffected by changes to a vector space basis (i.e., a coordinate rotation) but may be affected by translations (as in relative speed). A change of a vector space basis changes the description of a vector in terms of the basis used but does not change the vector itself, while a scalar has nothing to do with this change. In classical physics, like Newtonian mechanics, rotations and reflections preserve scalars, while in relativity, Lorentz transformations or space-time translations preserve scalars. The term "scalar" has origin in the multiplication of vectors by a unitless scalar, which is a *uniform scaling* transformation.

Examples of Scalar

Some examples of scalar include:

- Mass
- Speed
- Distance
- Time
- Volume
- Density
- Temperature

Victors:

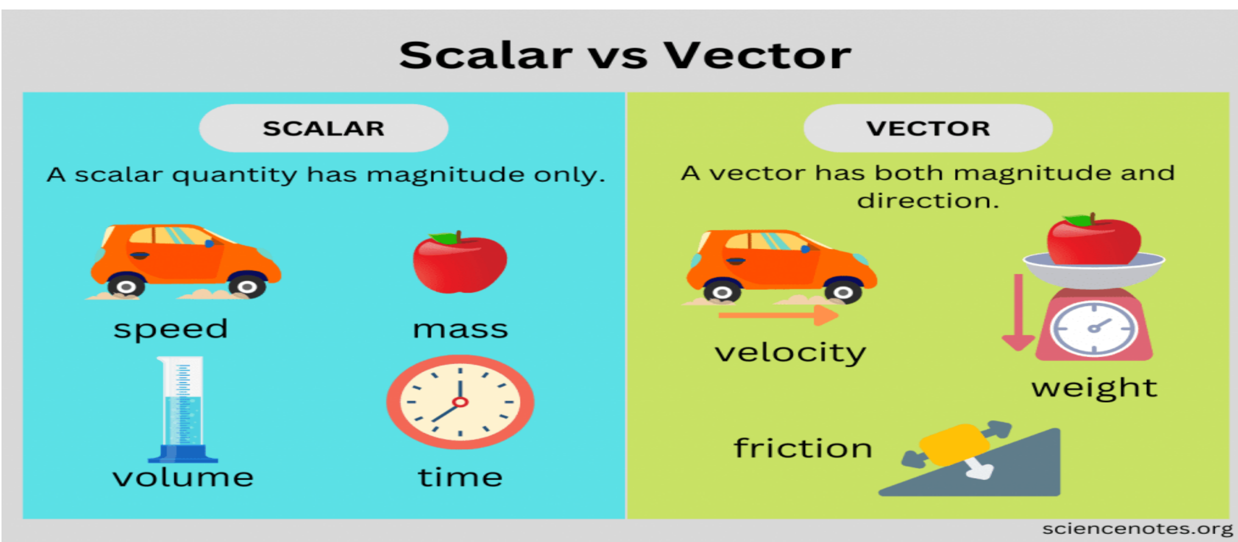
A vector quantity is defined as the physical quantity that has both directions as well as magnitude.

Examples of Vector

Examples of vector quantity include:

- Linear momentum
- Acceleration
- Displacement
- Momentum
- Angular velocity
- Force
- Electric field
- Polarization

The difference between Scalars and Vectors is crucial to understand



We have listed the various differences between a scalar and vector in the table below:

Scalar	**Vector**
Indicates magnitude or size	Indicates magnitude and direction
Examples include mass, temperature, speed, and time	Examples include velocity, acceleration, weight, and friction
May or may not have units (e.g., degrees, kilograms)	May be written in bold text or with an arrow over it to distinguish from scalar. Direction may be positive/negative or other (e.g., east, left, down).
Mathematical operations involving scalars produce another scalar. Operations on a scalar and a vector produce a vector.	Operations involving vectors produce a scalar or a vector. The dot product of two vectors is a scalar. Summation, subtraction, or the cross product of two vectors is a vector.

Distance and Spheres in Space

The formula for the **distance between two points** in the xy-plane extends to points $p_1(x_1, y_1, z_1)$ and $p_2(x_2, y_2, z_2)$

In space is:

$$|p_1p_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The **Midpoint Formula**

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

The **standard equation for the Sphere** of radius a and center $p(x_0, y_0, z_0)$ is given by

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2 .$$

Example: Find an equation of the sphere that has a diameter with endpoints $(3,-2,1)$ and center $(1,0,3)$.

Solution:

Using the distance formula with $p_1(3,-2,1)$ and $p_2(1,0,3)$ we find that the required distance is

$$|p_1p_2|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}=\sqrt{(1-3)^2+(0+2)^2+(3-1)^2}=\sqrt{4+4+4}=\sqrt{12}=2\sqrt{3}$$

so the radius of the sphere is $\frac{1}{2}(2\sqrt{3}) or \sqrt{3}$. The midpoint of the line segment joining $p_1(3,-2,1)$ and $p_2(1,0,3)$ is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)=\left(\frac{1+3}{2},\frac{0-2}{2},\frac{3+1}{2}\right)=(2,-1,1)$$

This point is the center of the sphere. Finally, we obtain the equation of the sphere:

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$
$$\Rightarrow (x-2)^2+(y+1)^2+(z-2)^2=3.$$

Example : Show that $x^2+y^2+z^2-4x+2y+6z+5=0$ is an equation of a sphere, and find its center and radius.

Solution:

By completing the squares in x, y and z, we can write the given equation in the form

$$x^2-4x+4-4+y^2+2y+1-1+z^2+6z+9-9+5=0$$
or
$$(x-2)^2+(y+1)^2+(z+3)=3^2$$

Comparing this equation with the equation of the sphere, we conclude that it is an equation of the sphere of radius 3 with center at $(2,-1,-3)$.

Vectors:

The vector represented by the directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q, and its **length** is denoted by $|\overrightarrow{PQ}|$.Two vectors are **equal** if they have the same length and direction.

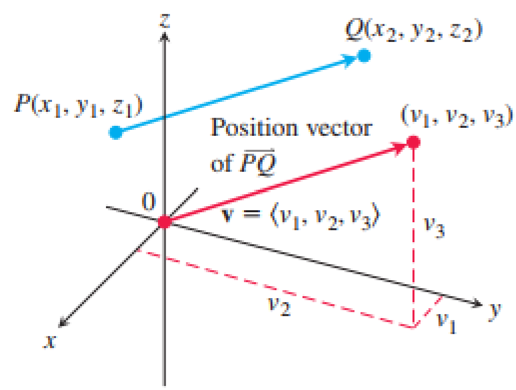
If **v** is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1,v_2), then the **component form** of v is

$v=\langle v_1,v_2\rangle$

If **v** is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1,v_2,v_3),then the **component form** of **v** is $v=\langle v_1,v_2,v_3\rangle$.

i.e/ given initial point $p(x_1,y_1,z_1)$ and terminal point $Q(x_2,y_2,z_2)$, the standard position vector $v=\langle v_1,v_2,v_3\rangle$ equal to $\overrightarrow{PQ}$ is

$$v=\langle x_2-x_1,y_2-y_1,z_2-z_1\rangle$$



. If **v** is two-dimensional with $p(x_1,y_1)$ and $Q(x_2,y_2)$ as points in the plane, then

$v=\langle x_2-x_1,y_2-y_1\rangle$

There is no third component for planar vectors.

Magnitude of a Vector

The **magnitude** or **length** of the vector **v** is the length of any of its representations and is denoted by the symbol $|v|\, or \,\|v\|$. By using the distance formula to compute the length of a segment $\overrightarrow{PQ}$, we obtain the following formulas.

The length of the two-dimensional vector $v = \langle v_1, v_2 \rangle$

$$|v| = \sqrt{(v_1)^2 + (v_2)^2}$$

The length of the three-dimensional vector $v = \langle v_1, v_2, v_3 \rangle$

$$|v| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} .$$

Example 1: Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$.

Solution:

(a) The vector $v = \overline{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \langle -5+3, 2-4, 2-1 \rangle = \langle -2,-2,1 \rangle$

(b) The **magnitude** or **length** of the vector **v** is

$$|v| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3 .$$

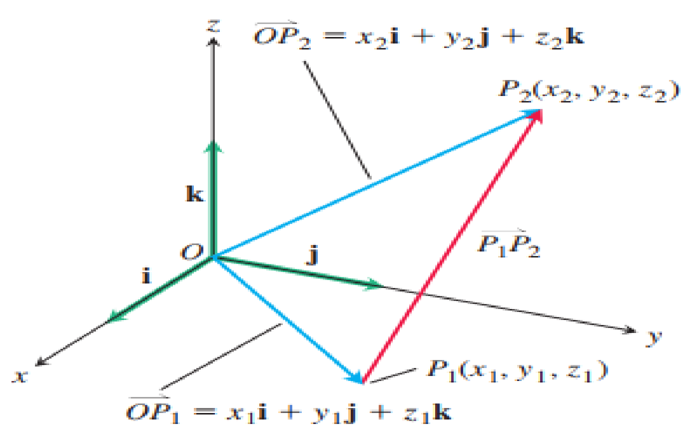
Unit Vectors

A vector **v** of length 1 is called a **unit vector**. The **standard unit vectors** are

$$i = \langle 1,0,0 \rangle, j = \langle 0,1,0 \rangle \; and \; k = \langle 0,0,1 \rangle$$

any vector $v = \langle v_1, v_2, v_3 \rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$v = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle$$
$$v = v_1 \langle 1,0,0 \rangle + v_2 \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle = v_1 i + v_2 j + v_3 k$$



We call the scalar (or number) y1 the of the vector v_1 the **i-component**, v_2 the **j-component** and v_3 the **k-component**. As shown in Figure the component form for the vector from $p_1(x_1, y_1, z_1)$ to $p_2(x_2, y_2, z_2)$ is

$$\overrightarrow{p_1p_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

If $v \neq 0$, then its length $|v|$ is not zero and

$$\left|\frac{v}{|v|}\right| = \frac{1}{|v|}|v| = 1$$

That is, $\frac{v}{|v|}$ is a unit vector in the direction of **v**, called **the direction** of the nonzero vector **v**.

Example 2: Find a unit vector u in the direction of the vector from $P(1,0,1)$ and $Q(3,2,0)$.

Solution:

We write $\overrightarrow{PQ}$ as a linear combination of the standard unit vectors and then divide it by its length:

$$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k = (3-1)i + (2-0)j + (0-1)k = 2i + 2j - k$$

$$|\overrightarrow{PQ}| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$u = \frac{\overrightarrow{PQ}}{|PQ|} = \frac{2i + 2j - k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

Example 3: Let $P(2,-1,2)$ and $Q(1,4,5)$ be two points in 3-space.

a. Find the vector V.

b. Find $|\overrightarrow{PQ}|$.

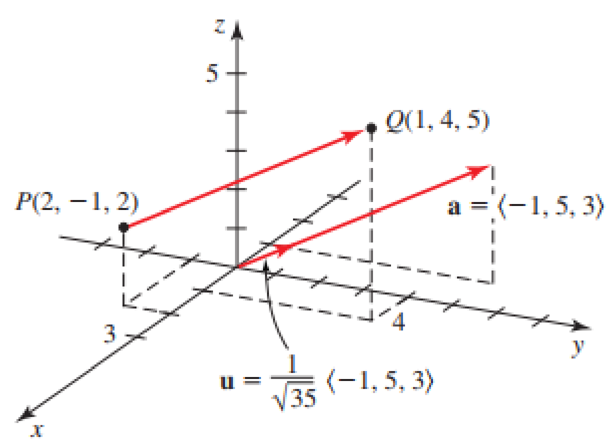


Figure: The vector $\overrightarrow{PQ}$, its equivalent position vector a, and the (position) unit vector **u.**

c. Find a unit vector having the same direction as $\overrightarrow{PQ}$.

Solution:

a. Using $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$, we have $\overrightarrow{PQ} = \langle 1-2, 4+1, 5-2 \rangle = \langle -1, 5, 3 \rangle$

b. Using the result of part (a), we have

$|\overrightarrow{PQ}| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} = \sqrt{(-1)^2 + (5)^2 + (3)^2} = \sqrt{35}$

c. Using the results of parts (a) and (b), we obtain the unit vector

$$u = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{-i + 5j + 3k}{\sqrt{35}} = \frac{-1}{\sqrt{35}} i + \frac{5}{\sqrt{35}} j + \frac{3}{\sqrt{35}} k$$

The vector $\overrightarrow{PQ}$, the position vector which equal a, and the unit vector u are shown in Figure.

Theorem: If $U = \langle u_1, u_2 \rangle$ and $V = \langle v_1, v_2 \rangle$ are vectors in 2-space and k is any scalar, then

$$U + V = \langle u_1 + v_1, u_2 + v_2 \rangle$$

$$U - V = \langle u_1 - v_1, u_2 - v_2 \rangle$$

$$kU = \langle ku_1, ku_2 \rangle$$

Similarly, if $U = \langle u_1, u_2, u_3 \rangle$ and $V = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space and k is any scalar, then

$$U + V = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$U - V = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

$$kU = \langle ku_1, ku_2, ku_3 \rangle$$

Example 4: if $u = \langle -2, 0, 1 \rangle$ and $v = \langle 3, 5, -1 \rangle$, then

$u+v=\langle 1,5,-4\rangle$

$3v=\langle -6,0,3\rangle$

$-u=\langle 2,0,-1\rangle$

$v-2w=\langle 3,5,-4\rangle-\langle -4,0,2\rangle=\langle 7,5,-6\rangle$

Properties of Vector Operations

Suppose **u**, **v**, and **w** are vectors and **a** and **c** are scalars. Then the following properties hold (for vectors in any number of dimensions)

1- $u+v=v+u$ Commutative property of addition

2- $(u+v)+w=u+(v+w)$ Associative property of addition

3- $v+0=0+v=v$ Additive identity

4- $v+(-v)=0$ Additive inverse

5- $c(u+v)=cu+cv$ Distributive property

6- $(a+c)v=av+cv$ Distributive property

7- $0v=0$ Multiplication by zero scalar

8- $c0=0$ Multiplication by zero vector

9- $1v=v$ Multiplicative identity

10- $a(cv)=(ac)v$ Associative property of scalar multiplication

Example 5: If $a=i+2j-3k$ and $b=4i+7k$ express the vector $2a+3b$ in terms of i,j and k.

Solution:

$$2a+3b=2(i+2j-3k)+3(4i+7k)=(2+12)i+4j+(-6+21)k=14i+4j+15k$$

1) Multiplying Two Vectors

(a) Dot Product

The **dot product** of two vectors $U = \langle u_1, u_2, u_3 \rangle$ and $V = \langle v_1, v_2, v_3 \rangle$ is given by

$$U.V = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$U.V$ is pronounced "U dot V."

Example 6: Find the dot product $P = \langle 2, -1, 2 \rangle$ and $Q = \langle 1, 4, 5 \rangle$.

Solution:

$$P.Q = 2*1 + (-1)*4 + 2*5 = 2 - 4 + 10 = 8.$$

Properties of The Dot Product

Suppose **U**, V, and W are vectors and **K** is a scalar. Then the following properties hold (for vectors in any number of dimensions)

1. $U.V = V.U$

2. $U.U = |U|^2$

3. $0.U = 0$

4. $K(U.V) = (KU).V = U.(KV)$

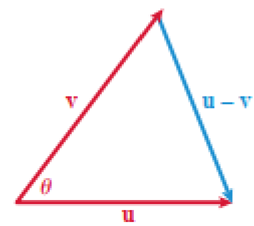
5. $(U + V).W = U.W + V.W$

6. $U.(V + W) = U.W + V.W$

Angle Between Two Vectors

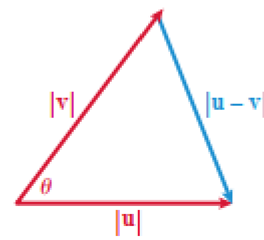
We can use these properties to develop an equation that relates the angle between two vectors and the dot product of the vectors.

Let **u** and **v** be two vectors with the same initial point, and let θ be the angle between them.



The vector $u-v$ is opposite angle θ.

A triangle is formed with side lengths equal to the magnitudes of the three vectors



Apply the Law of Cosines.

$$|u-v|^2 = |u|^2 + |v|^2 - 2|u||v|\cos(\theta)$$

Use properties of the dot product to rewrite the left side of equation

$|u-v|^2 = (u-v)(u-v)$ Property (2)

$= u.(u-v) - v(u-v)$ Property (6)

$= uu - uv - uv + vv$ Property (2)

$= |u|^2 + |v|^2 - 2uv$ Property (1)

Substitute this last expression for the left side of the original Law of Cosines equation.

$$|u|^2 + |v|^2 - 2uv = |u|^2 + |v|^2 - 2|u||v|\cos(\theta)$$

Simplify

$$uv = |u||v|\cos(\theta)$$

Isolate $\cos(\theta)$

$$\cos(\theta) = \frac{uv}{|u||v|}$$

Notice that **u** and **v** have to be nonzero vectors, since we divided by them in the last step.

Dot Product and Angles

1. The angle between two nonzero vectors **u** and **v** is $\theta = \cos^{-1}\left(\frac{uv}{|u||v|}\right)$.
2. The dot product of two vectors **u** and **v** is given by $uv = |u||v|\cos(\theta)$.

Example 7: Find the angle between $u = i - 2j - 2k$ and $v = 6i + 3j + 2k$

Solution:

$$u.v = 1*6 + (-2)*3 + (-2)*2 = -4$$

$$|u| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|v| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{uv}{|u||v|}\right) = \cos^{-1}\left(\frac{-4}{3*7}\right) \approx 1.76 \text{ Radians or } 100.98^\circ$$

Orthogonal Vectors

Vectors **u** and **v** are orthogonal if $u.v = 0$

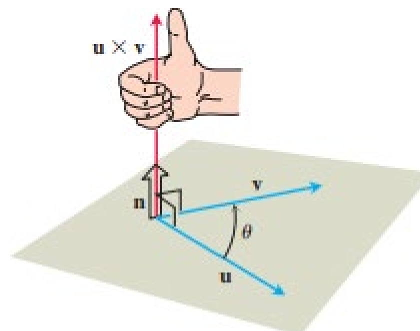
Example 8: Show that $2i + 3j - k$ is perpendicular to $5i - 4j + 2k$.

Solution:

Since

$$(2i+3j-k)(5i-4j+2k)=2(5)+2(-4)+(-1)(2)=0$$

these vectors are perpendicular.



(Orthogonal) Projection of u onto v

The **orthogonal projection of** u **onto** v, denoted

$$proj_v u = |u|\cos(\theta)\left(\frac{v}{|v|}\right),$$ where $|v| \neq 0$, is

The orthogonal projection may also be computed with the formulas

$$proj_v u = scal_v u\left(\frac{v}{|v|}\right)=\left(\frac{u.v}{v.v}\right)v,$$

where the **scalar component of** u **in the direction of** v is

$$scal_v u = |u|\cos(\theta)=\frac{u.v}{|v|}.$$

Example 9: Find the scalar projection and vector projection of $v=\langle 1,1,2\rangle$ onto $u=\langle -2,3,1\rangle$.

Solution:

Since $|u|=\sqrt{(-2)^2+3^3+1^2}=14$,

The scalar projection of v onto u is $scal_u v=\frac{u.v}{|u|}=\frac{(-2)(1)+(3)(1)+(1)(2)}{\sqrt{14}}=\frac{3}{\sqrt{14}}$.

The vector projection is this scalar projection times the unit vector u in the direction u.

$$proj_u v = scal_u v\left(\frac{u}{|u|}\right)=\frac{1}{\sqrt{14}}\left(\frac{\langle -2,3,1\rangle}{\sqrt{14}}\right)=\frac{3}{14}\langle -2,3,1\rangle=\left\langle -\frac{3}{7},\frac{9}{14},\frac{1}{14}\right\rangle.$$

(b) The Cross Product

1. The Cross product of Two Vectors in Space

The **cross product** $u \times v$ **("u cross v")** is the vector

$$u \times v = \left(|u||v|\sin(\theta)\right) n$$

2. Calculating the Cross product as a Determinant

If $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$ are vectors in 3-space, then the **cross-product u × v** is the vector defined by

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

or, equivalently,

$$u \times v = (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k$$

3. parallel Vectors

Nonzero vectors u and v are parallel if and only if $v \times u = 0$.

Theorem

Let U and V be nonzero vectors in 3-space. Then is $U \times V$ orthogonal to both U and V.

Example10: Find a vector perpendicular to the plane of $P(1,-1,0), Q(2,1,-1)$ and $R(-1,-1,2)$

Solution:

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k$$

$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= (2*2-(2)(-1))i - (1*2-(-1)(-2))\,j + (1*(2)-2*(-2))k = 6i + 6k$$

Properties of The Cross Product

If u,v and w are any vectors and r,s are scalars, then

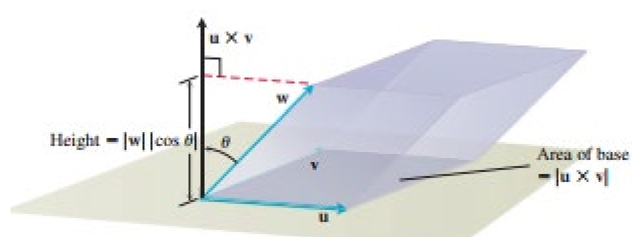
1. $(ru)\times(sv)=(rs)(u\times v)$
2. $u\times(v+w)=u\times v+u\times w$
3. $v\times u=-(u\times v)$
4. $(v+w)\times u=v\times u+w\times u$
5. $0\times u=0$
6. $u\times(v\times w)=(u.w)v-(u.v)w$

Triple Scalar or Box Product

The product $(u\times v).w$ is called the **triple scalar product** of u,v and w (in that order). As you can see from the formula

$(u\times v).w=|u\times v||w|\cos(\theta)$

the absolute value of this product is the volume of the parallelepiped (parallelogram-sided box) determined by u,v and w (Figure). The number $|u\times v|$ is the area of the base parallelogram. The number $|w|\cos(\theta)$ is the parallelepiped's height. Because of this geometry $(u\times v).w$ is also called the **box product** of u,v and w.



The triple scalar product can be evaluated as a determinant:

$$(u\times v).w=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Note: If we use the formula **triple scalar product** and discover that the volume of the parallelepiped determined by u,v and w is 0, then the vectors must lie in the same plane; that is, they are **coplanar**.

Example11: Find the volume of the box (parallelepiped) that is determined by $u=i+2j-k, v=-2i+3k$

and $w=7j-4k$.

Solution:

Using the rule for calculating a 3×3 determinant, we find

$$(u\times v).w=\begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}=\left((0)(-4)-3+7\right)*1-\left((-2)(-4)-3*0\right)*2+\left((-2)(7)-0*0\right)(-1)=-21-16+14=-23$$

The volume is $(u\times v).w=23$ units cubed.

Example12: Use the scalar triple product to show that the vector $u=\langle 1,4,-7\rangle, v=\langle 2,-1,4\rangle$, and w=$\langle 0,-9,18\rangle$ are coplanar.

Solution:

$$(u\times v).w=\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}=\left((-1)(18)-(4)(-9)\right)*1-\left((2)(18)-4*0\right)*4+\left((2)(-9)-(-1)*0\right)(-7)$$

$$=(18)-4(36)-7(18)=0$$

the volume of the parallelepiped determined by u,v and w is 0; this means that they are **coplanar**.

Direction Cosines

We can describe the direction of a nonzero vector a by giving the angles α, β and γ that makes with the positive x-, y-, and z- axes, respectively. (See Figure) These angles are called the **direction angles** of a The cosines of these angles $\cos(\alpha), \cos(\beta)$, and $\cos(\gamma)$ are called the **direction cosines** of the vector a.

Let $a = a_1 i + a_2 j + a_2 k$ be a nonzero vector in 3-space. Then

$$a.i = (a_1 i + a_2 j + a_2 k).i = a_1 \quad i.i = 1,\, j.i = k.i = 0$$

So

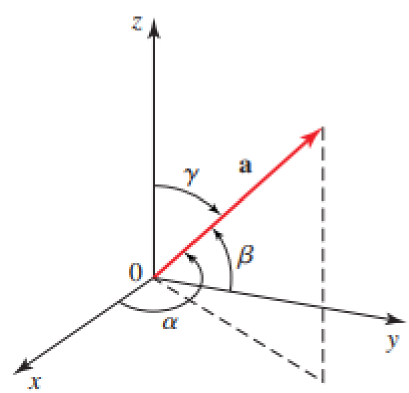
$$\cos(\alpha) = \frac{a.i}{|a||i|} = \frac{a_1}{|a|}$$

Similarly,

$$\cos(\alpha) = \frac{a_2}{|a|} \text{ and } \cos(\alpha) = \frac{a_3}{|a|}$$

By squaring and adding the three direction cosines, we obtain

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = \frac{a_1^2}{|a|^2} + \frac{a_2^2}{|a|^2} + \frac{a_3^2}{|a|^2} = 1.$$



Theorem:

The three direction cosines of a nonzero vector. $a = a_1 i + a_2 j + a_2 k$ in 3-space are

$$\cos(\alpha) = \frac{a_1}{|a|}, \cos(\beta) = \frac{a_2}{|a|} \text{ and } \cos(\gamma) = \frac{a_3}{|a|}$$

The direction cosines satisfy

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = \frac{a_1^2}{|a|^2} + \frac{a_2^2}{|a|^2} + \frac{a_3^2}{|a|^2} = 1$$

Note: If $a = a_1 i + a_2 j + a_2 k$ be a nonzero vector in 3-space, then the unit vector having the same direction
as is

$$u = \frac{a}{|a|} = \frac{a_1}{|a|} i + \frac{a_1}{|a|} j + \frac{a_1}{|a|} k = \cos(\alpha) i + \cos(\beta) j + \cos(\gamma) k$$

This shows that the direction cosines of a are the components of the unit vector in the direction of a . This augments the statement made earlier that the direction cosines of a vector define the direction of that vector.

Example 13: Find the direction angles of the vector $a = 2i + 3j + k$.

Solution:

We have

$$|a| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

So

$$\cos(\alpha) = \frac{a_1}{|a|} = \frac{2}{\sqrt{14}} \quad \cos(\beta) = \frac{a_2}{|a|} = \frac{3}{\sqrt{14}} \text{ and } \cos(\gamma) = \frac{a_3}{|a|} = \frac{1}{\sqrt{14}}$$

Therefore

$$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ \quad \beta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ \text{ and } \gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right).$$

Exercise:

Q1) Determine whether the given points are collinear.

1. $A(2,3,1), B(-4,0,5)$, and C(4,4,1).
2. $A(-1,3,-2), B(2,1,-1)$, and C(8,-3,1).

Q2) $a=\langle 1,-3,2\rangle$, b=$\langle -2,4,1\rangle$, and c=$\langle 2,-4,1\rangle$. Find the indicated quantity.

1. $a.(b+c)$
2. $(2a+3b).(3c)$
3. $(a.b)c-(b.c)a$
4. $(a-b).(a+b)$
5. $\left(\frac{a.b}{b.b}\right)b$
6. $|a-b|^2+|a+b|^2$

Q3) A vector has direction angles $\alpha=\frac{\pi}{3}$ and $\gamma=\frac{\pi}{4}$ Find the direction angle β.

Q4) find $proj_a b$ and $proj_b a$.

1. $a=\langle -3,4,-2\rangle, b=\langle 0,1,0\rangle$
2. $a=\langle 1,2,0\rangle, b=\langle -3,0,-4\rangle$
3. $a=2i+j+4k, b=3i+k$
4. $a=-i+2j, b=-3i+4j$

Q5) Find the value(s) of c such that the angle between $a=i+cj+2k$ and $b=-i+2j-k$ is 60°.

Q6) Let $u=\langle \sqrt{3},1,0\rangle, v=\langle 1,\sqrt{3},0\rangle$ and w=$\langle 1,\sqrt{3},2\sqrt{3}\rangle$

a. Compute $u.v$.

b. Find the angle between **u** and **v**.

c. Find the angle between **u** and **w**.

Q7) Let a and b be real numbers. Find all vectors $\langle 1,a,b\rangle$ orthogonal to $\langle 4,-8,2\rangle$.

Q8) Compute $|u\times v|$ if $|u|=3$ and $|v|=4$ and the angle between **u** and **v** is $\frac{2\pi}{3}$.

Q10) Let $u=5i-j+k, v=j-5k, w=-15i+3j-3k$. Which vectors, if any, are **(a)** perpendicular? **(b)** Parallel? Give reasons for your answers.

Q11) Find the area of the triangle with vertices $p(1,-1,0), Q(2,1,-1)$ and $R(-1,1,2)$.

Q12) For the given points P, Q, and R, find the approximate measurements of the angles of ΔPQR.

1- $P(1,-4), Q(2,7). R(-2,2)$

2- $P(0,-1,3), Q(2,2,1), R(-2,2,4)$

Let [u=5i-j+k,v=j-5k,w=-15i+3j-3k\] .

Which vectors, if any, are **(a)** perpendicular?

(b) Parallel? Give reasons for your answer