Also, $\{a_i\}$ closed $\forall\, i = 1,2,\dots,n$

$\Rightarrow\ \{a_i\}^c$ is open $\forall\, i = 1,2,\dots,n$

$\Rightarrow\ \bigcup_{i=1}^{n}\{a_i\}$ is open

$\Rightarrow\ \forall\, A_i \subset X,\ A_i$ is open

$\Rightarrow\ (X,\tau)$ is discrete

Definition (3.3): (T_2-space (Hausdorff Space)) فضاء هاوزدورف

We say that (X,τ) is **T_2-space** iff $\forall\, x,y \in X, x \neq y$, $\exists$ two disjoint open sets G,H, such that $x \in G, y \notin G\ \wedge\ y \in H, x \notin H$.

Example (3.10): Let $X = \{a,b\}$, $\tau = \{\emptyset,\{a\},\{b\},X\}$

We have (X,τ) is T_2-space. For

We have $a,b \in X, a \neq b$

$\exists$ two disjoint open sets $G = \{a\}, H = \{b\}$ with $a \in G, b \in H, \{a\}\cap\{b\} = \emptyset$

Remark (3.2): From the above definition, we conclude that $T_2 \Rightarrow T_1 \Rightarrow T_0$

But $T_1 \not\Rightarrow T_2$ as in the following example:

Example (3.11): Let $X = R$, $\tau = \{\emptyset, E \subset R : E^c\ is\ finite\}$.

Solution: We have see that (X,τ) is T_1-space

We need to show that (X,τ) is not T_2-space

Assume that (X,τ) is T_2-space

$\forall\, x,y \in X\ \ (x \neq y), G,H\ \ with\ G\cap H = \emptyset, x \in G, y \in H$

But $G\cap H = \emptyset$

$\Rightarrow\ G \subset H^c$

$\Rightarrow$ Contradiction [since G is an infinite set and H^c is finite]

Therefore (X,τ) is not T_2-space.

Theorem (3.5):

(1) The property (X, τ) T_2-space is hereditary.

(2) The property (X, τ) T_2-space is topological property.

Proof:

(1) Let (X, τ) be a T_2-space

We need to show that (X^*, τ^*) is T_2-space

Let $x, y \in X^*$, $x \neq y$

But $X^* \subset X$

$\Rightarrow\ x, y \in X\ \ (x \neq y)$

Since (X, τ) is T_2-space

$\Rightarrow\ \exists$ two disjoint open sets $G, H \in \tau$, $x \in G, y \in H$

Since $(X^*, \tau^*) \subset (X, \tau)$

$\Rightarrow\ \tau^* = \{G^* = G \cap \tau^* : G \in \tau\}$

We have $G \in \tau\ \Rightarrow\ G^* = G \cap X^*$ is in τ^*

$H \in \tau\ \Rightarrow\ H^* = H \cap X^*$ is in τ^*

Since $x \in G \cap X^*\ \Rightarrow\ x \in G^*$

$y \in H \cap X^*\ \Rightarrow\ y \in H^*$

We have $G^* \cap H^* = (G \cap X^*) \cap (H \cap X^*)$

$= (G \cap H) \cap X^* = \emptyset \cap X^* = \emptyset$

Hence, $\forall\ x, y \in X^*$, $x \neq y$, $\exists$ two disjoint open sets $G^*, H^* \in \tau^*$ s.t.

$x \in G^*\ \land\ y \in H^*$

$\Rightarrow\ (X^*, \tau^*)$ is a T_2-space

$\Rightarrow\ T_2$-space is a hereditary property.

(2) Let $f: (X, \tau) \to (X^*, \tau^*)$ be a homeo.

Let (X, τ) be T_2-space

We need to show that (X^*, τ^*) be T_2-space

Let $x^*, y^* \in X^*\ \ (x^* \neq y^*)$

Since f is onto

$\exists\ x, y \in X$ s.t. $x^* = f(x)$, $y^* = f(y)$

Since f is (1-1) $\Rightarrow\ x^* \neq y^* \ \Rightarrow\ x \neq y$

$\Rightarrow\ x, y \in X\ \ (x \neq y)$ and (X, τ) is T_1-space

$\Rightarrow\ \exists$ two disjoint G, H s.t. $x \in G, y \in H$

Since f is open

$\Rightarrow\ f(G) = G^*, f(H) = H^*$ is open in (X^*, τ^*)

And $G^* \cap H^* = f(G) \cap f(H) = f(G \cap H) = f(\emptyset) = \emptyset$

$\Rightarrow\ G^* \cap H^* = \emptyset$

Now, $x \in G\ \ \Rightarrow\ \ f(x) \in f(G)\ \ \Rightarrow\ \ x^* \in G^*$

$y \in H\ \ \Rightarrow\ \ f(y) \in f(H)\ \ \Rightarrow\ \ y^* \in H^*$

$\therefore\ \exists$ two disjoint open sets $G^*, H^* \in \tau^*$ s.t. $x \in G^*\ \wedge\ y \in H^*$

$\Rightarrow\ (X^*, \tau^*)$ is T_2-space

$\Rightarrow\ \ T_2$-space is a topological property.

Theorem (3.6): A compact set in a T_2-space is closed.

Proof: Let (X, τ) be a T_2-space

Let $A \subset X$ be compact

We need to show that A is closed, that is A^c open

We show that A^c is open

Let $x \in A^c\ \Rightarrow\ x \neq y, \forall\ y \in A$

We have $x, y \in X\ \ (x \neq y)$ and (X, τ) is T_2-space

$\Rightarrow\ \exists$ two disjoint open sets G_x, H_y s.t. $x \in G_x, y \in H_y$

We have the family of open sets $\{H_y : y \in A\}$ is an open cover of A $\{A \subset \bigcup_{y \in A} H_y\}$

But A is compact

$\Rightarrow\ H_{y_1}, H_{y_2}, H_{y_3}, \dots, H_{y_n}$ form a finite subcover of A

$\Rightarrow\ A \subset \bigcup_{i=1}^{n} H_{y_i}$

Now, let $G = \bigcap_{i=1}^{n} G_{x_i}$ and $H = \bigcup_{i=1}^{n} H_{y_i}$

We have $G = \bigcap_{i=1}^{n} G_{x_i} \subset A^c$ and $x \in G$

$\Rightarrow\ x \in G \subset A^c$

$\forall\ x \in A^c, \exists^{open}\ G \ni x$ and $x \in G \subset A^c$

$\Rightarrow\ A^c$ open $\Rightarrow\ A$ closed

Corollary (3.3): Let (X, τ) is a compact topological space and (X^*, τ^*) is T_2-space. If $f: (X, \tau) \to (X^*, \tau^*)$ is a continuous, one-to-one and onto function. Then f is topological homeomorphism.

Proof:

Let $f: (X, \tau) \to (X^*, \tau^*)$ be continuous, one-to-one and onto function.

Let (X, τ) be compact and (X^*, τ^*) be T_2-space

Since f is continuous, one-to-one and onto function, so to prove that f is homeo. we need to show that f is open

Let $G \subset X$ be any open set

$\Rightarrow\ G^c \subset X$ is closed

But (X, τ) is compact

$\Rightarrow\ G^c \subset X$ is compact

$\Rightarrow\ f(G^c) \subset X^*$ is compact

Since (X^*, τ^*) is T_2-space

$\Rightarrow\ f(G^c)$ closed

$\Rightarrow\ f(G)$ is open (since $f(G^c) = \left(f(G)\right)^c$ closed)

$\Rightarrow\ f$ is open

$\Rightarrow\ f$ is homeo.

Theorem (3.7): Every metric space is T_2-space.

Proof:

Let (M, d) be a metric

Let $x, y \in M, x \neq y$