We have $G = \bigcap_{i=1}^{n} G_{x_i} \subset A^c$ and $x \in G$

$\Rightarrow\ x \in G \subset A^c$

$\forall\ x \in A^c, \exists^{open}\ G \ni x$ and $x \in G \subset A^c$

$\Rightarrow\ A^c$ open $\Rightarrow\ A$ closed

Corollary (3.3): Let (X, τ) is a compact topological space and (X^*, τ^*) is T_2-space. If $f: (X, \tau) \to (X^*, \tau^*)$ is a continuous, one-to-one and onto function. Then f is topological homeomorphism.

Proof:

Let $f: (X, \tau) \to (X^*, \tau^*)$ be continuous, one-to-one and onto function.

Let (X, τ) be compact and (X^*, τ^*) be T_2-space

Since f is continuous, one-to-one and onto function, so to prove that f is homeo. we need to show that f is open

Let $G \subset X$ be any open set

$\Rightarrow\ G^c \subset X$ is closed

But (X, τ) is compact

$\Rightarrow\ G^c \subset X$ is compact

$\Rightarrow\ f(G^c) \subset X^*$ is compact

Since (X^*, τ^*) is T_2-space

$\Rightarrow\ f(G^c)$ closed

$\Rightarrow\ f(G)$ is open (since $f(G^c) = \big(f(G)\big)^c$ closed)

$\Rightarrow\ f$ is open

$\Rightarrow\ f$ is homeo.

Theorem (3.7): Every metric space is T_2-space.

Proof:

Let (M, d) be a metric

Let $x, y \in M, x \neq y$

$x \neq y \;\Rightarrow\; d(x,y) = \varepsilon > 0$

Consider two open sets (balls) $B_{\frac{\varepsilon}{3}}(x)$, $B_{\frac{\varepsilon}{3}}(y)$

To show that $B_{\frac{\varepsilon}{3}}(x) \cap B_{\frac{\varepsilon}{3}}(y) = \emptyset$

Assume that $B_{\frac{\varepsilon}{3}}(x) \cap B_{\frac{\varepsilon}{3}}(y) \neq \emptyset$

$\Rightarrow\; \exists\, z \in M$ and $z \in B_{\frac{\varepsilon}{3}}(x) \cap B_{\frac{\varepsilon}{3}}(y)$

$\Rightarrow\; z \in B_{\frac{\varepsilon}{3}}(x) \;\wedge\; z \in B_{\frac{\varepsilon}{3}}(y)$

$\Rightarrow d(x,z) < \frac{\varepsilon}{3} \;\wedge\; d(y,z) < \frac{\varepsilon}{3}$

Now, $d(x,y) \leq d(x,z) + d(z,y)$

$\Rightarrow\; \varepsilon < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} \;\Rightarrow\; \varepsilon < \frac{2\varepsilon}{3}$ contradiction

$\therefore\; B_{\frac{\varepsilon}{3}}(x) \cap B_{\frac{\varepsilon}{3}}(y) = \emptyset$

Then (M,d) is T_2-space

Theorem (3.8): In T_2-space, it is possible to separate, any point and a compact subset that does not contain the point by disjoint open sets.

Proof:

Let (X,τ) be a T_2-space

Let $E \subset X$ be compact

Let $x \in X$, $x \notin E$ be any point

But $x \notin E \;\Rightarrow\; x \neq y, \forall\, y \in E$

Since (X,τ) is T_2-space and $x, y \in X$ $(x \neq y)$

$\Rightarrow\; \exists$ two disjoint open sets G_x, H_y such that $x \in G_x \;\wedge\; y \in H_y$

We have $E \subset \bigcup_{y \in E} H_y$

$\Rightarrow\; \{H_y : y \in E\}$ is an open cover of E

Since E is a compact set, then $\exists$ a finite subcover of E s.t. $E \subset \bigcup_{i=1}^{n} H_{y_i} = H$

$\Rightarrow\; E \subset H$

Since $G_x \cap H_{y_i} = \emptyset,\; \forall\, i = 1,2\,, \dots, n$

i.e. $(G_x \cap H_{y_1}) \cup (G_x \cap H_{y_2}) \cup \dots (G_x \cap H_{y_n}) = \emptyset$

$\Rightarrow \; G_x \cap H = \emptyset$

$\Rightarrow \; E \subset H \;\wedge\; x \in G_x$ and $G_x \cap H = \emptyset$

Definition (3.4): If τ and τ^* two topologies defined on X $\{i.e. (X,\tau)$ and (X,τ^*) are topological spaces$\}$. We say that τ is أنعم **finer (smaller)** than τ^*, denoted by $\tau \leq \tau^*$, if each element of τ is an element of τ^*, τ^* is أخشن **coarser** than τ.

Example (3.12): If $\tau = \{\emptyset, X\}$ weak and $\tau^* = \{\emptyset, A \subset X\}$ discrete. τ is finer than τ^*.

Example (3.12): If (X,τ) is T_i-space, $\forall\, i = 0,1,2$ and τ is finer than τ^*, then (X,τ^*) is T_i-space, $\forall\, i = 0,1,2$.

Solution:

Let (X,τ) be T_0-space

$\Rightarrow \; \forall\, x,y \in X,\, x \neq y,\, \exists^{open}\, G$ s.t. $x \in G, y \notin G$

Since $\tau \leq \tau^*$, that is every element of τ is an element of τ^*

$\Rightarrow \; G \in \tau^*$ and $\forall\, x,y \in X,\, x \in G, y \notin G$

$\Rightarrow \; (X,\tau^*)$ is T_0-space

Let (X,τ) is T_1-space $\Rightarrow \; \forall\, x,y \in X,\, (x \neq y)$

$\exists\, G,H\; s.t. x \in G, y \notin G \;\wedge\; y \in H, x \notin H$

Since $\tau \leq \tau^*$

$\Rightarrow \; G,H \in \tau^*$ and $\forall\, x,y \in X, x \in G, y \notin G \wedge y \in H, x \notin H$

$\Rightarrow \; (X,\tau^*)$ is T_1-space

Let (X,τ) is T_2-space $\Rightarrow \; \forall\, x,y \in X\; (x \neq y)$

$\exists$ two disjoint open sets G,H s.t. $x \in G, y \notin G \wedge y \in H, x \notin H$

Since $\tau \leq \tau^* \;\Rightarrow\; G,H \in \tau^*, \forall\, x,y \in X, x \in G, \wedge\, y \in H$, s.t. $G \cap H = \emptyset$

$\Rightarrow \; (X,\tau^*)$ is T_2-space

Definition (3.5): Let (X, τ) be a topological space. Let $\{x_n\}_{n \in N}$ be a **sequence** of a points of X. We say that $\{x_n\}_{n \in N}$ convergence to $x \in X$ iff $\forall^{open}\, G \ni x$, $\exists\, N(G) \in \mathbb{N}, x_n \in G, \forall\, n \geq N(G)$.

Example (3.12): We have (N, τ) with $\tau = \{\emptyset, N, \{n, n+1, n+2, ...\}\}$ is topological space and $\{x_n\}_{n \in N}$ is sequence of points of $N = X$. According to the above definition we see that $\{n\} \to 1$, $\{n\} \to 2$, … , $\{n\} \to n$, $\forall\, n \in N$.

Theorem (3.9): In T_2-space, the convergent sequences converges to a unique point.

Proof:

Let (X, τ) be T_2-sapce

We need to show that $\{x_n\}$ be a converges to a unique point.

Assume that $\{x_n\}$ converges to $x\ \wedge\ \{x_n\}$ converges to $y\ (x \neq y)$

Now, $x, y \in X, x \neq y$ and (X, τ) is T_2-space

$\Rightarrow\ \exists$ two disjoint open sets $G, H; x \in G \wedge y \in H$

Now, $(x_n \to x)\ \Rightarrow\ \forall^{open}\, G \ni x, \exists\, N_1 \in \mathbb{N}, x_n \in G, \forall\, n \geq N_1$

$(x_n \to y)\ \Rightarrow\ \forall^{open}\, H \ni y, \exists\, N_2 \in \mathbb{N}, x_n \in H, \forall\, n \geq N_2$

Let $N = max(N_1, N_2)$

$\forall\, G \ni x, \exists\, N \in \mathbb{N}, x_n \in G, \forall\, n \geq N$

$\forall\, H \ni y, \exists\, N \in \mathbb{N}, x_n \in H, \forall\, n \geq N$

$\Rightarrow\ x_n \in G \cap H$, $\forall^{open}\, G \ni x$, $\forall^{open}\, H \ni y$

$\Rightarrow\ G \cap H \neq \emptyset$ Contradiction

$\therefore\ \{x_n\} \to \{x\}$ unique

Remark (3.3): The converse of the above theorem is not true is general

That is: $(\{x_n\} \to x$ (unique)) in $(X, \tau)\ \not\Rightarrow\ (X, \tau)$ is T_2-space.

Example (3.13): Let $X = R$, $\tau = \{\emptyset, E \subseteq R, E^c \; is \; finite\}$

The topological space (X, τ) is T_1-space

$x, y \in X, x \neq y$

$\{x\}$ finite $\Rightarrow$ $\{x\}^c \in \tau \;\wedge\; y \in \{x\}^c, x \notin \{x\}^c$

$\{y\}$ finite $\Rightarrow$ $\{y\}^c \in \tau \;\wedge\; x \in \{y\}^c, y \notin \{y\}^c$

The topological space is not T_2-space

Every sequence in (R, τ) if convergent it converges to a unique point.
