Chapter Four
الفضاءات المنتظمة والفضاءات السّوية
Regular and Normal Spaces

Definition (4.1): (Regular Space) الفضاء المنتظم

We say that (X, τ) is **regular space** denoted by $[R]$ if $\forall^{closed} F \subset X, \forall x \in X, x \notin F, \exists$ disjoint open sets G, H with $F \subset G \wedge x \in H$.

Example (4.1): Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Discuss whether (X, τ) is $[R]$ or not.

Solution: The closed sets are: $X, \{b, c\}, \{a\}, \emptyset$

(i) $F = \{b, c\}$ and $a \notin F$, we have $G = \{b, c\}$ and $H = \{a\}$ are disjoint open set with $F \subset G \wedge a \in H$

(ii) $F = \{a\}$ and $b, c \notin F$

We have $G = \{a\}$ and $H = \{b, c\}$ are disjoint open set with $F \subset G \wedge b \in H$

Also, $F \subset G \wedge c \in H$

Hence, $\forall^{closed} F \subset X, \forall x \notin F, \exists$ disjoint open sets G, H with $F \subset G \wedge x \in H$

$\Rightarrow$ (X, τ) is $[R]$

Theorem (4.1):

(1) The property (X, τ) is $[R]$ is hereditary.

(2) The property (X, τ) is $[R]$ is a topological property.

Proof:

(1) Let (X, τ) be $[R]$

Let (X^*, τ^*) be a topological subspace of (X, τ)

We need to show that (X^*, τ^*) is $[R]$

Let F^* be closed set in X^*

Let $x \in X^*$ and $x \notin F^*$

We have $F^* = F \cap X^*$

$x \in X^*, x \notin F^* \ \Rightarrow \ x \notin F$

Now, F closed set in X, $x \notin F$ and (X, τ) is $[R]$

$\exists$ disjoint open sets G, H s.t. $F \subset G \wedge x \in H$

We have $G^* = G \cap X^*$, $H^* = H \cap X^*$

Since G, H are open sets in (X, τ)

$\Rightarrow \ G^*, H^*$ are open set in (X^*, τ^*)

Now, $G^* \cap H^* = (G \cap X^*) \cap (H \cap X^*)$

$= (G \cap H) \cap X^* = \emptyset \cap X^* = \emptyset$

Since $x \in X^*, x \in H \ \Rightarrow \ x \in H^*$

and $F \subset G$

$\Rightarrow \ F \cap X^* \subset G \cap X^*$

$\Rightarrow \ F^* \subset G^*$

$\Rightarrow \ (X^*, \tau^*)$ is $[R]$

$\Rightarrow \ [R]$ is a hereditary property.

(2) Let $f: (X, \tau) \rightarrow (X^*, \tau^*)$ be a homeo.

Let (X, τ) is $[R]$

Let $F^* \subset X^*$ be a closed subset, and $x^* \in X^*$ such that $x^* \notin F^*$

Since f is continuous

$\Rightarrow \ F = f^{-1}(F^*)$ is a closed set in X (i.e. $f(F) = F^*$)

Since f onto and $x^* \in X^*$

$\exists \ x \in X$, such that $f(x) = x^*$

Since f is $(1-1)$ and $x^* \notin F^*$

$\Rightarrow \ f(x) \notin F^*$

$\Rightarrow \ x \notin f^{-1}(F^*) = F$

Now, F closed set in X, $x \notin F$ and (X, τ) is $[R]$

$\Rightarrow \ \exists$ disjoint open sets G, H s.t. $F \subset G \wedge x \in H$

Since f is open

$\Rightarrow$ $G^* = f(G)$ and $H^* = f(H)$ are open in X^*

$G^* \cap H^* = f(G) \cap f(H)$

$= f(G \cap H) = f(\emptyset) = \emptyset$

Now, $F \subset G \;\Rightarrow\; f(F) \subset f(G) \;\Rightarrow\; F^* \subset G^*$

$x \in H \;\Rightarrow\; f(x) \in f(H) \;\Rightarrow\; x^* \in H^*$

$\Rightarrow$ $\forall$ closed sets F^* in X^*, and $\forall\; x^* \notin F^*$,

$\exists$ disjoint open sets G^*, H^* with $F^* \subset G^* \wedge x^* \in H^*$

$\Rightarrow$ (X^*, τ^*) is $[R]$

$\Rightarrow$ $[R]$ is a topological property.

Definition (4.2): ($\boldsymbol{T_3}$- Space)

We say that (X, τ) is $\boldsymbol{T_3}$**-space** if (X, τ) is T_1-space and $[R]$.

Example (4.2): Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, \{b\}, X\}$

Discuss whether (X, τ) is T_3-space or not.

Solution: The closed sets are: $X, \{b\}, \{a\}, \emptyset$

(i) $F = \{b\}$ and $b \notin F$, we have $G = \{b\}$ and $H = \{a\}$ are disjoint open set with $F \subset G \wedge a \in H$

(ii) $F = \{a\}$ and $b \notin F$

We have $G = \{a\}$ and $H = \{b\}$ are disjoint open set with $F \subset G \wedge b \in H$

Hence, $\forall^{closed}\, F \subset X, \forall\, x \notin F$, $\exists$ disjoint open sets G, H with $F \subset G \wedge x \in H$

$\Rightarrow$ (X, τ) is $[R]$

Also, (X, τ) is T_1-space, because for $a, b \in X$, $a \neq b$ and $\exists^{open}\, G = \{a\}, H = \{b\}$; $a \in G, b \notin G \;\wedge\; a \notin H, b \in H$

$\Rightarrow$ (X, τ) is T_3-space
