Lecture (2)

- Some remarks on the c.d.f of continuous r.v.
- Mathematical expectation
- Properties with examples

Remark:- If X is a continious r.v. , then its c.d.f , $F(x)$ is everywhere continious and the derivative of F with respect to x exists and is equal to $f(x)$ at each point of continuity of $f(x)$. If X is a discrete r.v. its p.m.f , $f(x)$, is not the derivative of $F(x)$ with respect to x as shown in this discussion.

Properties of the c.d.f. :-

Let $F(x)$ be the c.d.f. of the r.v. X , then

i) $F(-\infty) = \lim\limits_{x \to -\infty} F(x) = 0$ and $F(+\infty) = \lim\limits_{x \to \infty} F(x) = 1$

ii) $F(x)$ is a monatone non-decreasing function of x , that is if $a < b$, then $F(a) \leq F(b)$.

iii) $F(x)$ is continious from the right, that is
$$\lim_{h \to 0^+} F(x+h) = F(x)$$

and $F(x)$ is continious everywhere if X is a continious r.v.

Ex(1):-

Let the r.v. X has a p.m.f given by

$$f(x) = p(X=x) = \begin{cases} 2k & x=0 \\ k & x=1 \\ 3k & x=2 \\ 0 & \text{o.w.} \end{cases}$$

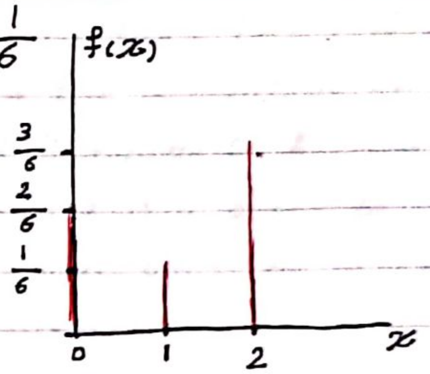
Find the (c.d.f) of X.

Sol:-

To find the (c.d.f) of X , we need to determine first the value of the constant k , and this can be acheived by using the fact that $\sum\limits_{\text{all } x} f(x) = 1$, then

$2k + k + 3k = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$

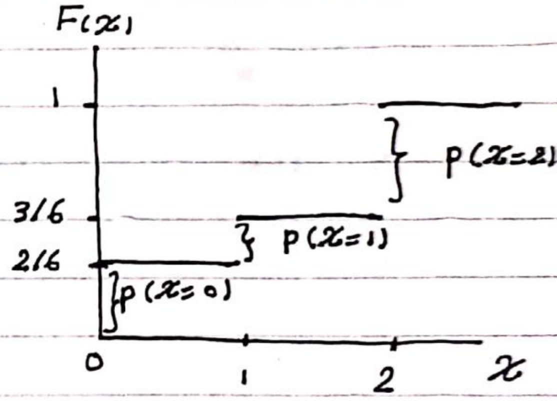
$$\therefore f(x) = \begin{cases} \frac{2}{6} & x = 0 \\ \frac{1}{6} & x = 1 \\ \frac{3}{6} & x = 2 \\ 0 & \text{o.w.} \end{cases}$$



The p.m.f of r.v. X

Then the c.d.f of X is:

$$F(x) = \begin{cases} 0 & X < 0 \\ \frac{2}{6} & 0 \leq x < 1 \\ 3/6 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$



The c.d.f of the r.v. X

Ex(2):-

Let X be a r.v. of the continious type with a p.d.f given by:-

$$f(x) = \begin{cases} \frac{2}{X^3} & 1 \leq X < \infty \\ 0 & \text{o.w.} \end{cases}$$

Find the c.d.f of X and find the probability statement such as $p(3 < X \leq 6)$.

Sol:-

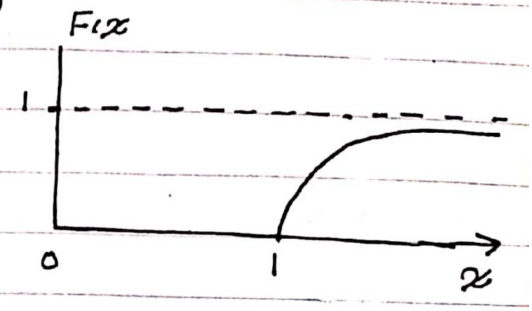
To find the c.d.f. of X, we have:

$F(x) = 0$ if $X < 1$ and if $X \geq 1$ then

$$F(x) = \int_1^x f(u)\,du = \int_1^x \frac{2}{u^3}\,du = \int_1^x 2u^{-3}\,du = 1 - \frac{1}{x^2}$$

Then the c.d.f of X is given by:-

$$F(x) = \begin{cases} 1 - \dfrac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$



The c.d.f. of X

To find a probability statement such as $p(3 < X \leq 6)$, we have

$$p(3 < X \leq 6) = F(6) - F(3)$$

$$= \left(1 - \frac{1}{36}\right) - \left(1 - \frac{1}{9}\right) = \frac{36-1}{36} - \frac{9-1}{9}$$

$$= \frac{35}{36} - \frac{8}{9} = \frac{35-32}{36} = \frac{1}{12}$$

Also this probabitity can be found by using the p.d.f as follows:-

$$p(3 < X \leq 6) = \int_3^6 f(x)\,dx = \int_3^6 \frac{2}{x^3}\,dx = 2\left[\frac{-1}{2x^2}\right]_3^6 = \frac{1}{12}$$

Remark:-

If X is a continious r.v. then:

$$p(a < X < b) = p(a \leq X < b) = p(a < X \leq b) = p(a \leq X \leq b)$$
$$= F(b) - F(a).$$

Ex(3):-

Let $f(x) = 1$ for $0 < x < 1$ and zero otherwise, be the p.d.f of the r.v. X. Define the r.v. $Y = \sqrt{X}$. Find $p(Y \leq y)$, where $0 < y < 1$.

Sol:-

To find this probability, we need the $dist^{n.}$ of

Y which is unknown, and hence we will use the distn. of X for that as follows:-

$$p(Y \leq y) = p(\sqrt{X} \leq y) = p(X \leq y^2)$$

$$= \int_0^{y^2} f(x)\,dx = \int_0^{y^2} dx = x \Big|_0^{y^2} = y^2 \qquad 0<y<1$$

But $p(Y \leq y) = F(y)$. From the definition of the c.d.f, therefore

$$F(y) = \begin{cases} 0 & y<0 \\ y^2 & 0\leq y<1 \\ 1 & y \geq 1 \end{cases}$$

And since Y is a continious r.v., the p.d.f of Y will be found by differentiating $F(y)$ with respect to y, that is:

$$f(y) = \begin{cases} 2y & 0<y<1 \\ 0 & o.w. \end{cases}$$

Ex(4):-

For the following probability distribution table find $F(-1)$, $F(0)$, $F(1)$, $F(2)$, $F(3)$

X	0	1	2	3
p(X)	$\frac{5}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$

Sol:-

$$F(-1) = \sum_{X=-\infty}^{-1} p(X) = 0$$

Ex (9) :-

IF $F(x) = \begin{cases} 0 & x < 0 \\ 5/12 & 0 \leq x < 1 \\ 7/12 & 1 \leq x < 2 \\ 11/12 & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$

Find $f(0)$, $f(1)$, $f(2)$, $f(3)$

Sol :-

$\because f(x_{i+1}) = F(x_{i+1}) - F(x_i)$

where $x_0 < x_1 < x_2 < \cdots$

$\therefore f(0) = F(0) - F(-1) = \frac{5}{12} - 0$

$f(1) = F(1) - F(0) = \frac{7}{12} - \frac{5}{12} = \frac{1}{6}$

$f(2) = F(2) - F(1) = \frac{11}{12} - \frac{7}{12} = \frac{1}{3}$

$f(3) = F(3) - F(2) = 1 - \frac{11}{12} = \frac{1}{12}$

Mathematical Expectation

Let X be a r.v. with a probability function f. Let $u(X)$ be a function of X such that

$\int_{-\infty}^{\infty} u(x) f_X(x)\, dx$ exists, if X is a continious r.v. with p.d.f f.

or $\sum_x u(x) f_X(x)$ exists, if X is a discrete r.v. with p.m.f f.

The integral, or the summation, is called the mathematical expectation of expected value of $u(x)$, and is denoted by $E[u(x)]$. That is:

$$E[u(x)] = \begin{cases} \int_{-\infty}^{\infty} u(x) f_X(x)\, dx & \text{if } X \text{ is a continious r.v.} \\ \sum_x u(x) f_X(x) & \text{if } X \text{ is a discrete r.v.} \end{cases}$$

The following are facts associated with the concept of expectation, which are:-

1- If K is a constant, then $E(K) = K$

2- If K is a constant and $u(x)$ is a function of X, then $E[K u(x)] = K E[u(x)]$

3- If K_1 and K_2 are two constants, $u_1(x)$ and $u_2(x)$ are functions of X, then
$E[K_1 u_1(x) + K_2 u_2(x)] = K_1 E[u_1(x)] + K_2 E[u_2(x)]$

4- Repeated application of fact (3) shows that if $K_1, K_2, ---, K_m$ are constants and $u_1(x), u_2(x), ---, u_m(x)$ are functions of X, then

$$E\left[\sum_{i=1}^{m} K_i u_i(x)\right] = \sum_{i=1}^{m} K_i E[u_i(x)]$$

Definition (1):-

The notation $X \sim f(x)$ will be used to indicate that the r.v. X has a probability function f or a distribution specified by its probability function f.

EX(1) :-

Let X has a probability function f such that

$$f(x) = \begin{cases} \frac{x}{6} & x = 1,2,3 \\ 0 & o.w. \end{cases}$$

Find $E(x)$, $E(x^2)$, $E(x+x^2)$ & $E(6x)$

Sol:-

$$E(x) = \sum_{x=1}^{3} x\, f_X(x)$$

$$= \sum_{x=1}^{3} x \cdot \frac{x}{6} = \frac{1}{6} \sum_{x=1}^{3} x^2 = \frac{1}{6}(1+4+9) = \frac{14}{6}$$

$$E(x^2) = \sum_{x=1}^{3} x^2 f_X(x) = \sum_{x=1}^{3} x^2 \cdot \frac{x}{6} = \frac{1}{6} \sum_{x=1}^{3} x^3$$

$$= \frac{1}{6}(1+8+27) = \frac{36}{6} = 6$$

$$E(x+x^2) = E(x) + E(x^2) = \frac{14}{6} + 6 = \frac{50}{6}$$

$$E(6x) = 6E(x) = 6\left(\frac{14}{6}\right) = 14$$

EX(2) :- واجب

$$\text{Let } X \sim f_X(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & o.w. \end{cases}$$

Find $E(x)$, $E(x^3)$, $E(\frac{1}{3}x)$ and $E(5x^3 - x)$

Sol:-

$$E(x) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^1 x \cdot 2(1-x)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{3}$$

$$E(x^3) = \int_{-\infty}^{\infty} x^3 f_X(x)\,dx = \int_0^1 2x^3(1-x)\,dx = 2\left[\frac{x^4}{4} - \frac{x^5}{5}\right]\Big|_0^1$$

$$= 2\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{1}{10}$$

$$E\left(\frac{1}{3}x\right) = \frac{1}{3}E(x) = \frac{1}{3}\left(\frac{1}{3}\right) = \frac{1}{9}$$

$$E(5x^3 - x) = 5E(x^3 - E(x) = 5\left(\frac{1}{10}\right) - \frac{1}{3} = \frac{1}{6}$$

EX(3):-

Let $X \sim f_X(x) = \begin{cases} \frac{1}{5} & 0 < x < 5 \\ 0 & \text{o.w.} \end{cases}$

Find $E(x)$, $E(5-x)$, $E[x(5-x)]$

Sol:-

$$E(x) = \int_0^5 x \cdot \frac{1}{5}\,dx = \frac{1}{10}x^2\Big|_0^5 = \frac{25}{10} = \frac{5}{2}$$

$$E(5-x) = 5 - E(x) = 5 - \frac{5}{2} = \frac{5}{2}$$

$$E(x(5-x)) = 5E(x) - E(x^2)$$

$$E(x^2) = \int_0^5 x^2 \cdot \frac{1}{5}\,dx = \frac{x^3}{15}\Big|_0^5 = \frac{25}{3}$$

Therefore $E[x(5-x)] = 5\left(\frac{5}{2}\right) - \frac{25}{3} = \frac{25}{6}$

Now, if we look at $E[x(5-x)]$ and compare it with $E(x) \cdot E(5-x)$, we get

$E[x(5-x)] = \frac{25}{6}$, while $E(x) \cdot E(5-x) = \frac{5}{2} \cdot \frac{5}{2} = \frac{25}{4}$ $\neq \frac{25}{6}$, It follows that $E[x(5-x)] \neq E(x)E(5-x)$.