Lecture (11)

- Conditional expectations
- Examples

(5)

Conditional expectation:-

Definition:-

Let (X,Y) be a two-dimensional r.v. and let $g(\cdot,\cdot)$ be a function of two variables. The conditional expectation of $g(X,Y)$ given $X=x$, denoted by:

$E[g(X,Y)|X=x]$, is defined to be

$$E[g(X,Y)|X=x]=\begin{cases}\sum_{y} g(x,y)\, f(Y|x) & \text{if } Y \text{ is a discrete type} \\ \int_{-\infty}^{\infty} g(x,y)\, f(y|x)\, dy & \text{if } Y \text{ is continuous type}\end{cases}$$

Similarly

$$E[g(x,y)|Y=y]=\begin{cases}\sum_{x} g(X,Y)\, f(X|Y) & \text{if } X \text{ is a discrete r.v.} \\ \int_{-\infty}^{\infty} g(X,Y)\, f(X|Y)\, dx & \text{if } X \text{ is a continuous r.v.}\end{cases}$$

EX:- Let (X,Y) : $(0,0)$, $(1,2)$, $(1,3)$ with

$f(X,Y)$: $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$

Find $E(Y|X=1)$, $E(Y^2|X=1)$ and $E[Y+\frac{1}{2}Y^2|X=1]$

Sol:-

$$E(Y|X=1) = \sum_{Y_K} Y_K \, f(Y|X=1)$$

$$f(Y|X) = \frac{f(x;y)}{f(x)}$$

X; 0 , 1

$f(x)$; $\frac{1}{3}$, $\frac{2}{3}$

$Y|X=1$; 2 3

$f(Y|X=1)$; $\frac{1}{2}$ $\frac{1}{2}$

$$f(Y_1|X=1) = \frac{f(1,2)}{f(X=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$f(Y_2|X=1) = \frac{f(1,3)}{f(X=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Therefore

$$E(Y|X=1) = \sum_{Y_K} Y_K \, f(Y|X=1) = 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) = \frac{5}{2}$$

$$E(Y^2|X=1) = \sum_{Y_K} Y_K^2 \, f(Y|X=1) = (2)^2\left(\frac{1}{2}\right) + (3)^2\left(\frac{1}{2}\right) = \frac{13}{2}$$

$$E(Y + \frac{1}{2}Y^2|X=1) = E(Y|X=1) + \frac{1}{2}E(Y^2|X=1)$$
$$= \frac{5}{2} + \frac{1}{2}\left(\frac{13}{2}\right) = \frac{5}{2} + \frac{13}{4} = \frac{10+13}{4} = \frac{23}{4}$$

ex:-

Let $(X,Y) \sim f(x,y) = \begin{cases} 1 & 0<X,Y<1 \\ 0 & \text{o.w.} \end{cases}$

Find $E(Y|X=x)$, $E(Y^2|X=x)$, $E(Y^2|X=\frac{1}{3})$

Sol:-

$$E(Y|X=x) = \frac{f(x,y)}{f(x)} \quad = \int_0^1 f(x,y)\,dy = \int_0^1 dy = y\Big|_0^1 = 1$$

$$\Rightarrow f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore f(y|X=x) = \begin{cases} 1 & 0<Y<1 \quad 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$E(Y|X=x) = \int_0^1 y\,dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E(Y^2|X=x) = \int_0^1 y^2\,dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$E(Y^2|X=\tfrac{1}{2}) = \int_0^1 y^2\,dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$$

Ex:- Let $(X,Y) \sim f(x,y) = \begin{cases} x+y & 0<x,y<1 \\ 0 & \text{o.w.} \end{cases}$

Find $E(Y|X=\frac{1}{2})$

Sol:-

$$E(Y|X=\tfrac{1}{2}) = \int_0^1 y\, f(Y|X=\tfrac{1}{2})\,dY$$

$$f(Y|X) = \frac{f(X,Y)}{f(X)}$$

$$f(X) = \int_0^1 f(x,y)\,dy = \int_0^1 (X+Y)\,dY = XY + \frac{Y^2}{2}\Big|_0^1$$

$$= X + \frac{1}{2}$$

$$\therefore f(X) = \begin{cases} X+\frac{1}{2} & 0<X<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore f(Y|X) = \begin{cases} \dfrac{X+Y}{X+\frac{1}{2}} & 0<Y<1 \quad 0<X<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore f(Y|X=\tfrac{1}{2}) = \begin{cases} \dfrac{Y+\frac{1}{2}}{\frac{1}{4}} & 0<Y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore E(Y|X=\tfrac{1}{2}) = \int_0^1 Y\, f(Y|X=\tfrac{1}{2})\,dY$$

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$$= \int_0^1 Y \cdot \frac{Y + \frac{1}{2}}{\frac{1}{4}} \, dY = 4 \left[\frac{Y^3}{3} + \frac{1}{2} \cdot \frac{Y^2}{2} \Big|_0^1 \right]$$

$$= 4 \left[\frac{Y^3}{3} + \frac{Y^2}{4} \right]_0^1 = \frac{7}{3}$$

Theorem:-

Let (X,Y) be a two-dimensional r.v. and $g_1(\cdot)$ and $g_2(\cdot)$ be functions of one variable. Then

1) $E_Y[g_1(Y) + g_2(Y) \mid X=x] = E[g_1(Y) \mid X=x] + E[g_2(Y) \mid X=x]$

2) $E_Y[g_1(Y)\, g_2(X) \mid X=x] = g_2(X)\, E_Y[g_1(Y) \mid X=x]$

Theorem:-

Let X, Y be two r.v's. having joint probability function $f(x,y)$ and marginal $f(x)$ and $f(y)$ respectively. Then X and Y are indep. iff:-

$f(X) = f(X|Y)$ for all X and Y.

proof:-

If X and Y are indep.

$f(X,Y) = f(X)\, f(Y)$. Therefore, for $f(Y) > 0$, we have:-

$$f(X|Y) = \frac{f(x,y)}{f(y)} = \frac{f(x)\, f(y)}{f(y)} = f(x).$$

$\therefore f(x) = f(x/y)$

Similarly, we can prove that if X and Y are independent then

$f(Y) = f(Y/X)$ for all X and Y.

Example:-

$$\text{Let } (X,Y) \sim f(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(Y/X=x)$, $E(XY|X=x)$ and $Var(Y|X=x)$

Sol:-

$$E(Y/X=x) = \int_{-\infty}^{\infty} Y f(Y|X)\, dY = \int_{X}^{1} Y f(Y|X)\, dY$$

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$f(x) = \int_{X}^{1} f(x,y)\, dy = \int_{X}^{1} 8XY\, dY = 8X \frac{Y^2}{2} \Big|_{X}^{1}$$

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$$= \begin{cases} 4X(1-X^2) & 0<X<1 \\ 0 & o.w. \end{cases}$$

$$\therefore f(y|X) = \frac{f(x,y)}{f(x)} = \frac{8XY}{4X(1-X^2)}$$

$$= \begin{cases} \frac{2Y}{1-X^2} & X<Y<1 \\ 0 & o.w. \end{cases}$$

$$\therefore E(Y|X=x) = \int_X^1 Y \cdot \frac{2Y}{1-X^2}\, dY = \frac{2}{1-X^2}\int_X^1 Y^2 dY$$

$$= \frac{2}{1-X^2}\left[\frac{Y^3}{3}\Big|_X^1\right] = \frac{2}{1-X^2}\left[\frac{1}{3} - \frac{X^3}{3}\right]$$

$$= \frac{2}{1-X^2} \cdot \frac{1-X^3}{3} = \begin{cases} \frac{2(1-X^3)}{3(1-X^2)} & 0<X<1 \\ 0 & o.w. \end{cases}$$

$$E(XY|X=x) = X\,E(Y|X=x)$$

$$= \begin{cases} \frac{2}{3} \cdot \frac{X(1-X^3)}{(1-X^2)} & 0<X<1 \\ 0 & o.w. \end{cases}$$

$$E(Y^2|X=x) = \int_X^1 Y^2 f(Y|X)\, dY = \int_X^1 Y^2 \cdot \frac{2Y}{1-X^2}\, dY$$

$$= \frac{2}{1-X^2}\int_X^1 Y^3 dY = \frac{2}{1-X^2}\left(\frac{Y^4}{4}\Big|_X^1\right)$$

$$= \begin{cases} \frac{1-X^4}{2(1-X^2)} & 0<X<1 \\ 0 & o.w. \end{cases}$$

$$Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2$$

$$= \begin{cases} \frac{1-X^4}{2(1-X^2)} - \frac{4}{9} \cdot \frac{(1-X^3)^2}{(1-X^2)^2} & 0<X<1 \\ 0 & o.w. \end{cases}$$

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Ex:-

Let $(X,Y) \sim f(x,y) = \begin{cases} e^{-(x+y)} & \\ 0 & o.w. \end{cases}$

a) Find the marginal and conditional distributions

b) Find $P(X>1)$, $P(0<X<1 \mid Y=2)$, $E(X+Y)$ and $E(XY)$.

c) Find the covariance and the correlation coefficient of X and Y, and then find the regression line of Y on X.

Sol:-

a) $f(x) = \int_0^\infty f(x,y)\,dy = \int_0^\infty e^{-(x+y)}\,dy = e^{-x}\int_0^\infty e^{-y}\,dy$

$= \begin{cases} e^{-x} & x>0 \\ 0 & o.w. \end{cases}$

$f(y) = \int_0^\infty f(x,y)\,dx = \int_0^\infty \exp-(x+y)\,dx = \begin{cases} e^{-y} & y>0 \\ 0 & o.w. \end{cases}$

Therefore

$f(X \mid Y) = \dfrac{f(X,Y)}{f(Y)} = \dfrac{e^{-x}e^{-y}}{e^{-y}} = \begin{cases} e^{-x} & x>0 \\ 0 & o.w. \end{cases}$

$f(Y \mid X) = \dfrac{e^{-(x+y)}}{e^{-x}} = \begin{cases} e^{-y} & y>0 \\ 0 & o.w. \end{cases}$

Since the conditional distributions are the same of the marginal distributions, this means X and Y are independent r.v's.

b) $p(X>1) = \int_1^\infty f(x)\,dx = \int_1^\infty e^{-x}\,dx = e^{-1}$

$p(0<X<1 \mid Y=2) = \int_0^1 f(X \mid Y=2)\,dx = \int_0^1 e^{-x}\,dx = 1-e^{-1}$

$E(X+Y) = E(X) + E(Y)$

$$= \int_0^\infty x e^{-x} dx + \int_0^\infty y e^{-y} dy$$

let $u = x \Rightarrow du = dx$

$dv = e^{-x} dx \Rightarrow v = \int e^{-x} dx = -e^{-x}$

$$\Rightarrow \int_0^\infty x e^{-x} = -x e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x} dx$$

$$= 0 + -e^{-x} \Big|_0^\infty = 1$$

$$\therefore \int_0^\infty y e^{-y} dy = 1 \quad \Rightarrow E(X+Y) = 1+1 = 2$$

$$E(XY) = \int_0^\infty \int_0^\infty xy e^{-(x+y)} dx dy$$

$$= \int_0^\infty y e^{-y} \left[\int_0^\infty x e^{-x} dx \right] dy = \int_0^\infty y e^{-y} dy = 1$$

c) $Cov(X,Y) = E(XY) - E(X) E(Y)$

$= 1 - (1)(1) = 0$

Note, this result can be obtained directly since X and Y are indep. as shown in part (a)

To find the regression curve of Y on X, we should note that it is equal to $E(Y|X=x)$. Therefore

$$E(Y|X=x) = \int_{-\infty}^{\infty} y f(y|X=x) dy = \int_0^\infty y e^{-y} dy = 1$$

That is the regression curve of Y on X is constant, it does not depend on the value of X (why?).

Ex:-

Suppose that X and Y are two jointly distributed discret r.v's. Let X has a Bernoulli distn with parameter p. That is, $p(X=1) = 1 = 1 - p(X=0)$. Suppose that

$E(Y|X=0) = 1$ and $E(Y|X=1) = 2$. What is $E(Y)$?

Sol:-

$E(Y) = E[E(Y|X=x)]$

$$= \sum_{X=0}^{1} E(Y|X=x) f(x)$$

$$= (1)(1-p) + 2(p) = 1 - p + 2p = 1 + p$$

Ex:-

The discrete density of X is given by:-

$$f(x) = \begin{cases} \frac{X}{3} & X = 1, 2 \\ 0 & o.w. \end{cases}$$

Let $f(Y|X)$ be a Binomial density with parameters X and $\frac{1}{2}$, that is

$$f(Y|X) = p(Y=y|X=x) = \begin{cases} \binom{x}{y} \left(\frac{1}{2}\right)^{y} \left(\frac{1}{2}\right)^{x-y} & y = 0, 1, \ldots x \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} \binom{x}{y} \left(\frac{1}{2}\right)^{x} & y = 0, 1, 2, \ldots x \\ 0 & o.w. \end{cases}$$

Find

a) $E(Y)$ b) The joint distn of X and Y

Sol:-

a) $E(Y) = E[E(Y|X=x)]$, $\sum_{Y} Y f(Y|X=x) = E(Y|X)$

$$= \underbrace{\sum_{Y=0}^{X} Y \binom{X}{Y} \left(\frac{1}{2}\right)^{X}}_{\text{The expected value of } Y \text{ having } Bin(X, \frac{1}{2})} = \frac{X}{2} \quad \text{a r.v.}$$

$\therefore E(Y) = E\left[\frac{1}{2} X\right] = \frac{1}{2} E(X) = \frac{1}{2} \sum x f(x)$

$= \frac{1}{2} \sum_{x=1}^{2} x \cdot \frac{x}{3} = \frac{1}{6} (1^2 + 2^2) = \frac{5}{6}$

6- To find the joint $dist^{n.}$ of X and Y, we have:

$$f(x,y) = f(y|x) f(x) = \begin{cases} \binom{x}{y} \left(\frac{1}{2}\right)^x \left(\frac{x}{3}\right) & , x = 1, 2, \\ & \;\; y = 1, 2, \ldots x \\ 0 & \text{o.w.} \end{cases}$$