Lecture (12)

- Joint moment generating function
- Some remarks and examples

Joint moment generating functions and moments:-

Definition:-

The joint moments of X and Y defined by $E(X^r Y^s)$, where r and s are any non-negative integers. The joint moments about the mean (joint central moments) are defined by $E(X-\mu_x)^r(X-\mu_y)^s$.

We have to point out the relation between joint moments of the r.v. (X,Y) and moments of either the r.v. X or the r.v. Y as follows:-

If $r=1$, $s=0$, then $E(X^1Y^0)=E(X)=\mu_x$. But if $r=0$, $s=1$, then $E(X^0Y^1)=E(Y)=\mu_y$.

Now, if $r=2$, $s=0$, or $r=0$, $s=2$, then we get $E(X^2Y^0)=E(X^2)$, $E(X^0Y^2)=E(Y^2)$.

which they are the second moments of X and Y

Therefore, $Var(X)$ and $Var(Y)$ can be found from these moments.

The mixed second moment $E(XY)$ is obtained when $r=1$, $s=1$, since: $E(X^1Y^1)=E(XY)$

And $cov(X,Y) = E(XY) - E(X)E(Y)$
Similarly, if we let $r=1$ and $s=0$, then $E(X-\mu_x)^1(Y-\mu_y)^0 = E(X-\mu_x) = 0$ and if we let $r=0$ and $s=1$, then $E(X-\mu_x)^0(Y-\mu_y)^1 = 0$
Now if $r=2$, $s=0$, then
$E(X-\mu_x)^2(Y-\mu_y)^0 = E(X-\mu_x)^2 = \sigma_x^2$, and if $r=0$ $s=2$, then $E(X-\mu_x)^0(Y-\mu_y)^2 = \sigma_y^2$
Therefore, we can obtain $Var(X)$ or $Var(Y)$ directly from the second pure central moments.
Finally, if $r=1$ and $s=1$, then
$E(X-\mu_x)^1(Y-\mu_y) = Cov(X,Y)$. That is, we obtain the covariance of X and Y directly from the second mixed central moment.

Ex:-
Let $(X,Y) \sim f(X,Y) = \begin{cases} 1 & 0<X<1, \ 0<Y<1 \\ 0 & o.w \end{cases}$

Find $E(X^rY^s)$ then find all pure and mixed moments.

sol:-

$$E(X^rY^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s f(X,Y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x^r y^s \,dx\,dy = \int_0^1 y^s\left[\frac{x^{r+1}}{r+1}\Big|_0^1\right]dy$$

$$= \frac{1}{r+1}\int_0^1 y^s\,dy = \frac{1}{r+1}\left[\frac{y^{s+1}}{s+1}\right]_0^1 = \frac{1}{(r+1)(s+1)}$$

Now, if $r=1$ and $s=0$, then $E(X)=\frac{1}{2}$

If $r=0$ and $s=0$, then $E(Y)=\frac{1}{2}$

But if $r=2$ and $s=0$, then $E(X^2)=\frac{1}{3}$

If $r=0$, $s=2$, then $E(Y^2)=\frac{1}{3}$

Therefore $Var(Y)=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$, $Var(X)=\frac{1}{12}$

Now if $r=1$ and $s=1$, then $E(XY)=\frac{1}{4}$. Then

$Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{2}=0$

Also $\rho_{X,Y}=0$ since $Cov(X,Y)=0$

<u>Definition:-</u>

The joint moment generating function of X and Y is defined by:-

$M_{X,Y}(t)=E(e^{t_1X_1+t_2X_2})$, assuming that the expectation exists for all values of t_1 and t_2 such that $-h_1<t_1<h_1$, $-h_2<t_2<h_2$ for some $h_1>0$, $h_2>0$.

Note that if $M_{X,Y}(t_1,t_2)$ exists, then the r-th moment of X may be obtained from $M_{X,Y}(t_1,t_2)$ by differentiating it r-th times with respect to t_1 then taking the limit as all t's approach to zero.

Also $E(X^r Y^s)$ can be obtained by differentiating the joint moment generating function r times with respect to t_1 and s times with respect to t_2 and taking the limit as all t's approach to zero.

This will be stated in the following theorem

theorem :-

If $M_{X,Y}(t_1,t_2)$ exists for t_1 and t_2 such that $-h_1 < t_1 < h_1$, $-h_2 < t_2 < h_2$, for some $h_1 > 0$, $h_2 > 0$. then

i) $M_X(t_1) = M_{X,Y}(t_1,0) = \lim_{t_2 \to 0} M_{X,Y}(t_1,t_2)$

and similarly

$M_Y(t_2) = M_{X,Y}(0,t_2) = \lim_{t_1 \to 0} M_{X,Y}(t_1,t_2)$

ii) $E(X^r) = \frac{\partial^r}{\partial t_1^r} M_{X,Y}(t_1,t_2) \Big|_{t_1=0,\, t_2=0}$

and similarly

$E(Y^s) = \frac{\partial^s}{\partial t_2^s} M_{X,Y}(t_1,t_2) \Big|_{t_1=0,\, t_2=0}$

iii) $E(X^r Y^s) = \frac{\partial^{r+s}}{\partial t_1^r \, \partial t_2^s} M_{X,Y}(t_1,t_2) \Big|_{t_1=0,\, t_2=0}$

Definition:-

The joint moment generating function of X and Y is defined by:-

$M_{X,Y}(t) = E(e^{t_1 X_1 + t_2 X_2})$, assuming that the expectation exists for all values of t_1 and t_2 such that $-h_1 < t_1 < h_1$, $-h_2 < t_2 < h_2$ for some $h_1 > 0$, $h_2 > 0$.

Note that if $M_{X,Y}(t_1, t_2)$ exists, then the r-th moment of X may be obtained from $M_{X,Y}(t_1, t_2)$ by differentiating it r-th times with respect to t_1, then taking the limit as all t's approach to zero.

Also $E(X^r Y^s)$ can be obtained by differentiating the joint moment generating function r times with respect to t_1 and s times with respect to t_2 and taking the limit as all t's approach to zero.

This will be stated in the following theorem

Corollary:-
If X and Y are indep., then $Cov(X,Y)=0$

proof:-

$Cov(X,Y) = E(XY) - E(X)E(Y)$

Since X and Y are indep., then $E(XY)=E(X)E(Y)$

$\therefore Cov(X,Y) = E(X)E(Y) - E(X)E(Y) = 0$

Definition:-
Two r.v's X and Y are defined to be uncorrelated if and only if $Cov(X,Y)=0$

theorem:-
If X and Y are indep., then $\rho_{X,Y}=0$

Remark:-
The converse of the above theorem is not always true, that is $Cov(X,Y)=0$ does not imply that X and Y are uncorrelated indep.

يعني اذا كان X و Y مستقلين فهما غير مترابطان لكن اذا كان X و Y
غير مترابطان ليس من الضروري اذ يكونا مستقلين.

Ex:-
Consider a r.v. (X,Y) have the following prob. function

$$f(x,y) = \begin{cases} \frac{1}{4} & (X,Y) = (-1,1), (1,1) \\ \frac{1}{2} & (x,y) = (0,0) \\ 0 & o.w. \end{cases}$$

Find $E(XY)$, $Cov(X,Y)$

Sol:-

Y \ X	-1	0	1	f(y)
0	0	$\frac{1}{2}$	0	$\frac{1}{2}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
f(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

then $f(x) = \sum_{all\,y} f(x,y) = \begin{cases} \frac{1}{4} & x=-1,1 \\ \frac{1}{2} & x=0 \\ 0 & o.w \end{cases}$

and $f(y) = \sum_{all\,x} f(x,y) = \begin{cases} \frac{1}{2} & y=0,1 \\ 0 & o.w \end{cases}$

Therefor X and Y are dependen var., since $f(x,y) \neq f(x)f(y)$

$E(X) = \sum_{x} X f(X) = \frac{1}{4}(-1) + \frac{1}{4}(1) + \frac{1}{2}(0) = 0$

$E(Y) = \sum_{Y} Y f(Y) = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$

$E(XY) = \sum_{X}\sum_{Y} XY f(X,Y) = -1(1)(\frac{1}{4}) + (1)(1)(\frac{1}{4})$

$+ (0)(0)(\frac{1}{2}) = -\frac{1}{4} + \frac{1}{4} = 0$

$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

Thus $Cov(X,Y) = 0$ but X and Y are dependent

Remark:-

The uncorrelated r.V's need not be indep. but indep. r.V's have to be uncorrelated.

theorem:-

If X and Y are two r.V's, then

$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

proof:-

$Var(X+Y) = E[(X+Y) - E(X+Y)]^2$

$= E[(X+Y) - E(X) - E(Y)]^2$

$= E[(X - \mu_X) + (Y - \mu_Y)]$

$= E(X-\mu_x)^2 + E(Y-\mu_y)^2 + 2E(X-\mu_x)(Y-\mu_y)$

$= Var(X) + Var(Y) + 2\,Cov(X,Y)$

Corollary

If X and Y are indep. r.v's then

$Var(X+Y) = Var(X) + Var(Y)$

theorem:-

Two jointly distributed r.v's X and Y are indep if and only if:-

$M_{X,Y}(t_1,t_2) = M_X(t_1)\, M_Y(t_2)$

for all t_1 and t_2 for which $-h_j < t_j < h_j$ for some $h_j > 0$, $j = 1, 2$.

proof:-

$M_{X,Y}(t_1,t_2) = E(e^{t_1 X + t_2 Y})$

$= E(e^{t_1 X}\, e^{t_2 Y})$

Since X and Y indep., then

$E(e^{t_1 X}\, e^{t_2 Y}) = E(e^{t_1 X})\, E(e^{t_2 Y})$

$= M_X(t_1)\, M_Y(t_2)$

theorem:-

If X and Y are indep r.v's, then

i) $M_{X+Y}(t) = M_X(t)\, M_Y(t)$

ii) $\Phi_{X+Y}(t) = \Phi_X(t)\, \Phi_Y(t)$

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