Lecture (6)

Moment generating function

Characteristic function

Theorems and examples

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الحث هنا

The moment generating function

Suppose that there is a positive number h such that $E(e^{tX})$ exists for some $-h<t<h$, then the function $M_X(t)$ defined by :-

$$M_X(t) = E(e^{tX}) \quad \text{for } -h<t<h$$

is called the moment generating function (m.g.f) of the r.v. X.

$$M_X(t) = E(e^{tX}) \begin{cases} \sum_{x=-\infty}^{\infty} e^{tx} f(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \end{cases}$$

theorem (1) :-

$$M_X(0) = 1$$

proof :-

$$M_X(0) = E(e^0) = E(1) = 1$$

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Remark:-

We know that the m.g.f for a discrete r.v. may be written as:-

$$M_X(t) = E(e^{tX}) = \sum_{X=-\infty}^{\infty} e^{tX} f(x)$$

Now $\boxed{e^{Y} = \sum_{k=0}^{\infty} \frac{Y^k}{k!}}$ بشكل عام

then $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$

Then substitute e^{tX} in the m.g.f, we have:-

$$M_X(t) = \sum_{X=-\infty}^{\infty} \left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots\right) f(x)$$

$$= \sum_{x=-\infty}^{\infty} \left[f(x) + tXf(x) + \frac{t^2X^2}{2!} f(x) + \cdots\right]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$$

Therefore, we can deduce that the r-th moment about the origion is the coefficient of $\frac{t^r}{r!}$ by using the Maclourian's series of e^{tX}.

theorem:-

$$\left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0} = \mu_r' \quad \text{for } r = 0, 1, 2, \ldots$$

proof:-

$\because$ m.g.f $= M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots$

$$\frac{dM_X(t)}{dt} = 0 + \mu_1' + t\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots$$

$$\Rightarrow \left.\frac{dM_X(t)}{dt}\right| = \mu_1'$$

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$$\frac{d^2 M_x(t)}{dt^2} = M_2' + t M_3' + \dots$$

$$\Rightarrow \left.\frac{d^2 M_x(t)}{dt^2}\right|_{t=0} = M_2'$$

Therefore in general, we have $\left.\frac{d^r M_x(t)}{dt^r}\right|_{t=0} = M_r'$

theorem:-

Given x as a r.v. and a as a constant then

$$M_{(ax)}^{(t)} = M_x(at)$$

proof:-

$$M_{ax}(t) = E(e^{tax}) = E(e^{(at)x} = M_x(at)$$

theorem:-

$$M_{x+a}(t) = e^{at} M_x(t)$$

proof:-

$$M_{x+a}(t) = E(e^{t(x+a)}) = e^{at} E e^{(t)x} = e^{at} M_x(t)$$

Generalization of these two theorems is given in the following theorem:-

theorem:-

$$M_{\frac{x+a}{b}}^{(t)} = e^{\frac{at}{b}} M_x\left(\tfrac{t}{a}\right)$$

proof:-

$$M_{\frac{x+a}{b}}(t) = E e^{t\left(\frac{x+a}{b}\right)} = e^{\frac{at}{b}} E e^{\frac{t}{b}x} = e^{\frac{at}{b}} M_x\left(\tfrac{t}{b}\right)$$

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Ex(1):-
Given the p.d.f of X by $f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & o.w \end{cases}$

Find the m.g.f of X

sol:-

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} e^{-x}\, dx = \int_0^\infty e^{-(1-t)x}\, dx$$

$$= -\frac{1}{1-t} e^{-(1-t)x}\Big|_0^\infty = \frac{1}{1-t} e^{-0} = \frac{1}{1-t} \qquad |t|<1$$

Ex(2):-
Assume X has the following probability distribution

x	0	1
$p(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

Find the m.g.f of X

sol:-

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{1} e^{tx} p(X=x) = e^{0} p(X=0) + e^{t} p(X=1)$$

$$= \frac{1}{2} + e^{t}\left(\frac{1}{2}\right) = \frac{1}{2}(1+e^{t})$$

Ex(3):-
In the ex(1) above, obtain M_1' and M_2' then use them to find σ^2.

sol:-

$$M_X(t) = \frac{1}{1-t}$$

From theorem, we have $\frac{d^r M_X(t)}{dt^r}\Big|_{t=0} = M_r'$

$$\Rightarrow M_1' = \frac{dM_X(t)}{dt}\Big|_{t=0} = -(1-t)^{-2}(-1)\Big|_{t=0} = 1$$

$$\mu_2' = \frac{d^2 M_x(t)}{dt^2}\Big|_{t=0} = \frac{d}{dt}(1-t)^{-2}\Big|_{t=0}$$

$$= -2(1-t)^{-3}(-1)\Big|_{t=0} = 2$$

$$\sigma^2 = E(X-\mu_1)^2 = E(X^2)-(EX)^2 = \mu_2' - (\mu_1')^2$$
$$= 2-1 = 1$$

Ex(4):-

Given f(x) as in Ex(2), Find μ_1', μ_2', σ^2

Sol:-

$$M_x(t) = \frac{1}{2}(1+e^t)$$

$$\mu_r' = \frac{d^r M_x(t)}{dt^r}\Big|_{t=0}$$

$$\frac{dM_x(t)}{dt} = \frac{1}{2}e^t \Rightarrow \mu_1' = \frac{dM_x(t)}{dt}\Big|_{t=0} = \frac{1}{2}$$

$$\frac{d^2 M_x(t)}{dt} = \frac{1}{2}e^t \Rightarrow \mu_2' = \frac{d^2 M_x(t)}{dt^2}\Big|_{t=0} = \frac{1}{2}$$

$$\Rightarrow \sigma^2 = \mu_2' - (\mu_1')^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

H.W.-

1) Given the p.d.f of X by $f(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $0<\theta<1$ and $x=0,1,2,\ldots,n$. Find $M_x(t)$ and then use it to find μ_1', μ_2' and σ^2.

2) Given the p.d.f of X is defined as:-

$$f(x) = \begin{cases} 2x & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

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Obtain the m.g.f of X and then use it to find μ_1', μ_2' and σ^2.

3) Consider the p.d.f of X is defined as:

$f(x) = \lambda e^{-\lambda x}$ $\quad x, \lambda > 0$

Find the m.g.f of X and use it to show that $\mu_1' = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$.

The characteristic function الدالة الوصفية او التمييزية

The characteristic function denoted by $\phi_X(t)$ exists for every distribution, and is defined as:-

$$\phi_X(t) = E(e^{itX}) = \begin{cases} \sum_{\text{all } x} e^{itx} f(x) & \text{if } X \text{ is a discrete r.v.} \\ \int_{-\infty}^{\infty} e^{itx} f(x)\,dx & \text{if } X \text{ is a continious r.v.} \end{cases}$$

where i is a complex number $i = \sqrt{-1}$, and t is an arbitrary real number. It can be shown that every distribution function has a unique characteristic function. Therefore to each characteristic function there corresponds a unique distribution function.

Remark:-

Let X be a r.v. having probability function f. The characteristic function $\phi_X(t)$ of a r.v. X is also called the Fourier transformation of X.

Remark We can write $\phi_X(t)$ as follows

$\phi_X(t) = Ee^{itX} = E(\cos(tX) + i\sin(tX))$

This definition can be proved as follows:-

By using Maclaurin series to e^{itX}, we get:-

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$$e^{itx} = \sum_{k=0}^{\infty} \frac{(itx)^k}{k!} = 1 + itx - \frac{(tx)^2}{2!} + \frac{(itx)^3}{3!} + \frac{(tx)^4}{4!} + \frac{(itx)^5}{5!} + \cdots$$

$$= \left(1 - \frac{(tx)^2}{2!} + \frac{(tx)^4}{4!} - \frac{(tx)^6}{6!} + \cdots\right) + i\left(tx - \frac{(tx)^3}{3!} + \frac{(tx)^5}{5!} + \cdots\right)$$

$$= \cos(tx) + i \sin(tx).$$

Thus $\phi_x^{(t)} = E(\cos(tx) + i \sin(tx))$

Properties of the characteristic function:-

We discuss the properties of C.F by the following theorems:-

1) $\phi_x^{(0)} = 1$

proof:-

$$\phi_x^{(0)} = E(\cos(tx) + i \sin(tx))\Big|_{t=0} = Ee^{itx}\Big|_{t=0} = 1$$

2) $\phi_x^{(t)}$ characterize the distribution function.

3) $|\phi_x^{(t)}| \leq 1$, i.e: $\phi_x^{(t)}$ is bounded ane therefore always exist

proof:-

$$|\phi_x^{(t)}| = |E(e^{itx})| = \left|\int e^{itx} f(x)\, dx\right|$$

$$\leq \int |e^{itx}||f(x)|\, dx = \int |e^{itx}| f(x)\, dx \quad \text{since } f(x) > 0$$

But $|e^{itx}| = |\cos tx + i \sin tx| = \sqrt{\cos^2 tx + \sin^2 tx}$
$= \sqrt{1} = 1$

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$$\Rightarrow |\phi_X(t)| = \int 1 f(x)\, dx = 1$$

4) $\phi_{aX+b}(t) = e^{ibt}\, \phi_X(at)$ where ~~a and~~ b are constants

For example

If $Y = aX$

$\phi_Y(t) = \phi_X(at)$,

If $Y = X - b \Rightarrow \phi_Y(t) = e^{-ibt}\, \phi_X(t)$.

5) Uniqueness theorem

If there is a characteristic function then there is a corresponding unique distribution function which has that characteristic function.

يعني اذا كان هناك دالة كثافة احتمال بدلالة X سوف يكون له

ايضا دالة مميزة وحيدة يعني $f(x) \longrightarrow \phi_X(t)$

<u>theorem :-</u>

If $E(X^k)$ exist , so does $E(X^j)$ for $j = 1, 2, \dots, k-1$

and $\phi_X(t) = 1 + itE(X) + \frac{(it)^2}{2!} E(X^2) + \dots + \frac{(it)^k}{k!} E(X^k)$

$+\, o(it^r) \rightarrow o(t)$ تكتب

where $o(u)$ is a function such that $\frac{o(u)}{u} \rightarrow 0$ as $u \rightarrow 0$

$$\phi_X'(t) = \frac{d\phi_X(t)}{dt} = iEX + \frac{2i^2 t}{2!} E(X^2) + \dots + \frac{k i^k t^{k-1}}{k!} E(X^k) + o(it^r)$$

$$\phi_X'(t) = \left.\frac{d\phi_X(t)}{dt}\right|_{t=0} = iE(X).$$

$$\Rightarrow \boxed{E(X) = \frac{1}{i}\,\phi'_X(0)}$$

$$\phi''_X(0) = i^2 E(X^2) \qquad \Rightarrow E(X^2) = \frac{1}{i^2}\,\phi''_X(0)$$

$$\Rightarrow E(X^r) = \frac{1}{i^r}\,\phi^{(r)}_X(0) \qquad r \in \mathbb{N}$$

Remark $M_X(t)$ و $\phi_X(t)$ العلاقة بين

$$\phi_X(t) = M_X(it)$$

proof:-

$$\phi_X(t) = Ee^{itX} = Ee^{(it)X} = M_X(it)$$

Example:-

If $M_X(t) = e^{2t+3t^2}$ Find $\phi_X(t)$ and then $\phi_Y(t)$, $E(Y)$, $E(Y^2)$, $Var(Y)$, where $Y = 2X - 4$

sol:-

$\because M_X(t) = e^{2t+3t^2}$ and $\phi_X(t) = M_X(it)$

$$\Rightarrow \phi_X(t) = e^{2it-3t^2}$$

$$\phi_Y(t) = \phi_{2X-4}(t) = e^{-4it}\,\phi_X(2t) = e^{-4it}\,e^{2i(2t)-3(2t)^2}$$

$$E(Y) = \frac{1}{i}\,\phi'_Y(0) \quad , \quad E(Y^2) = \frac{1}{i^2}\,\phi''_Y(0)$$

$$\therefore Var(Y) = EY^2 - (EY)^2$$

Ex:- Let $f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ $\quad x = 0, 1, 2, \ldots$ Find $\phi_X(t)$, $M_X(t)$, $E(X^2)$

sol:-

$$\phi_X(t) = E(e^{itX}) = \sum_{x=0}^{\infty} e^{itx}\,\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x}{x!}$$

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