Lecture (5)

- The mode
- Median
- Examples

Definition :-

A mode of a distribution of one random variable X is a value of x that maximizes the p.d.f or p.m.f

Mode of a continious distribution

The mode of a r.v. X having continious distribution with p.d.f $f(x)$ is the value that makes $f(x)$ maximum. If the r.v. X has the mode at m, then to find m ; m must satisf

$$f'(x)\Big|_{X=m} = 0 \quad \text{and} \quad f''(x)\Big|_{x=m} < 0$$

Mode in a discrete distribution

The mode of a discrete distribution or discrete r.v. is the value taken by the r.v. tha corresponds to the largest probability. If ther are two values of the r.v. that corresponds to the same largest probability, then these two values are modes. That is, there are two modes in this case.

Definition:-

A median of a distribution of one r.v. X of the discrete or continious type is a value of x such that ~~$p(X < x) \leq \frac{1}{2}$~~

$$p(X \leq x) = p(X \geq x) = \frac{1}{2}$$

هذا يعني أن الوسيط هو تلك القيمة من قيم المتغير .
العشوائي X التي تقسم المساحة تحت المنحنى الى قسمين
نصف الاحتمال او المساحة الى اليمين والنصف الآخر الى
اليسار .

Ex 1- If X has the following probability density function

$$f(x) = \frac{2^x e^{-2}}{x!} \qquad x = 0, 1, 2, \ldots$$

Find the mode of this distribution.

Sol:-

We need the probability distribution for some points X

x:	0	1	2	3	4	5	6 ...
$f(x)$:	e^{-2}	$2e^{-2}$	$2e^{-2}$	$\frac{4}{3}e^{-2}$	$\frac{2}{3}e^{-2}$	$\frac{4}{15}e^{-2}$	$\frac{4}{45}e^{-2}$

We conclude

$$p(0) < p(1) = p(2) > p(3) > p(4) > \ldots$$

This means that the distribution has two modes which they are

$x_1 = 1$ and $x_2 = 2$ and the maximum value for the probability density function at these two modes will be

$$\max f(x)\Big|_{x_1=1} = \max f(x)\Big|_{x_2=2} = 2e^{-2}.$$

Ex:-

If you have the following probability density function
$f(x) = \frac{1}{2} x^2 e^{-x}$, $x > 0$

- Find the modes of the distribution.

sol:-

$f'(x) = \frac{1}{2} e^{-x} (-x^2 + 2x)$

$f'(x) = 0 \Rightarrow \frac{1}{2} e^{-x} (-x^2 + 2x) = 0 \Rightarrow e^{-x} x (x-2) = 0$

$\Rightarrow x_1 = 0$ or $x - 2 = 0 \Rightarrow x_2 = 2$ or $e^{-x} = 0 \Rightarrow x_3 \to \infty$

Now we test $f''(x)$ at these three points of x

For $x = 0 \Rightarrow f''(x)\Big|_{x=0} = \frac{1}{2} e^{-x} (x^2 - 4x + 2)\Big|_{x=0} > 0$

This solution is not accepted

For $x \to \infty \Rightarrow f''(x)\Big|_{x \to \infty} = \lim_{x \to \infty} f''(x) = 0$

Thus $x \to \infty$ is not accepted

but For $x = 2 \Rightarrow f''(x)\Big|_{x=2} = -e^{-x} < 0$

Thus the mode of this distribution is $x = 2$
and the maximum value of $f(x)$ at $x = 2$ is
$\max f(x)\Big|_{x=2} = 2e^{-2}$

Ex:- If $f(x) = e^{-x}$ $x > 0$ represents the p.d.f of a r.v. , X . Find the median of the distribution.

Sol:-

$M = \int_0^M e^{-x} dx = \frac{1}{2} \Rightarrow -e^{-x}\Big|_0^M = \frac{1}{2}$

$1 - e^{-M} = \frac{1}{2} \Rightarrow e^{-M} = (2)^{-1} \Rightarrow \Rightarrow M = -\ln(\frac{1}{2}) = 0.693$

Ex:- Given the p.d.f of X by $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & o.w. \end{cases}$

Find $\mu = EX$, μ_1' , μ_2' , μ_1 , μ_2 , σ^2 , σ

Sol:-

$$\mu = EX = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x(2x)\, dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$\mu_1' = E(X) = \frac{2}{3}$$

$$\mu_2' = E(X^2) = \int_0^1 x^2 (2x)\, dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$$

$$\mu_1 = E(X - \mu) = E(X) - \mu = \mu - \mu = 0$$

$$\mu_2 = E(X - \mu)^2 = \sigma^2 = E(X^2) - (EX)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

$$\sigma^2 = \mu_2 = \frac{1}{18}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{1}{18}}$$

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