Lecture (3)

- Some special mathematical expectation

- Examples

Remark:-

From Ex(3), we can conclude that in general, the expected value of a product of functions is not equal to the product of the expected value of the some functions.

Ex(4):- واجب

Consider the following probability distribution table:-

x	0	1	2
$f(x)$	5/12	3/12	4/12

Find the expected value of X

Sol:-

$$EX = \sum_{x} f(x).x \quad \text{or} \quad \sum_{x} x\,p(x)$$

$$= \sum_{x=0}^{2} x\,p(x) = 0\,p(X=0) + 1\,p(X=1) + 2\,p(X=2)$$

$$= 0 + 1\left(\frac{3}{12}\right) + 2\left(\frac{4}{12}\right) = \frac{11}{12}$$

Ex(5):- واجب

Given a r.v. X with a p.d.f $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$

Find E(X)

Sol:-

$$E(X) = \int_{x} x\,f(x)\,dx = \int_{0}^{\infty} x\,e^{-x}\,dx$$

let $u = x \Rightarrow du = dx$

and $dv = e^{-x}\,dx \Rightarrow v = \int e^{-x}\,dx = -e^{-x}$

$\int u\,dv = uv - \int v\,du$

$$\Rightarrow \int_0^\infty x e^{-x}\,dx = -x e^{-x}\Big|_0^\infty - \int_0^\infty -e^{-x}\,dx$$

$$= -x e^{-x}\Big|_0^\infty - e^{-x}\Big|_0^\infty$$

$$= -\infty e^{-\infty} + 0 e^{-0} - e^{-\infty} + e^{0} = 0 + 0 - 0 + 1 = 1$$

Ex (6)

Given a r.v. with a p.d.f $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$

Show that $E(X^r) = \dfrac{2}{(r+1)(r+2)}$ and use it to

evaluate $E[(2X+1)^2]$

Sol:-

$$E(X^r) = \int_x X^r f(x)\,dx$$

$$= \int_0^1 X^r \cdot 2(1-X)\,dx = 2\int_0^1 (X^r - X^{r+1})\,dx$$

$$= 2\left[\int_0^1 X^r dx - \int_0^1 X^{r+1} dx\right] = 2\left[\frac{X^{r+1}}{r+1} - \frac{X^{r+2}}{r+2}\right]\Big|_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right] = 2\left[\frac{r+2-(r+1)}{(r+1)(r+2)}\right] = \frac{2}{(r+1)(r+2)}$$

Now:-

$$E[(2X+1)^2] = E[(4X^2+4X+1)] = 4E(X^2)+4E(X)+1$$

$$= 4\cdot\frac{2}{(2+1)(2+2)} + 4\cdot\frac{2}{(1+1)(1+2)} + 1 = \frac{8}{12} + \frac{8}{6} + 1$$

$$= \frac{36}{12} = 3$$

H.W:-

Given $f(x) = \begin{cases} 2e^{-x} & x>0 \\ 0 & o.w. \end{cases}$

Find $E(X)$, $E(X^2)$, $E(2X+3X^2)$.

Ex(7):- واجب

Consider the following p.m.f:-

$$f(x) = \begin{cases} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{1-x} & x=0,1 \\ 0 & o.w. \end{cases}$$

Find $E(X)$, $E(2X+3)$, $E(X^2)$, $E(X^3)$,

$E(1+2X+3X^2+5X^3)$

Sol:-

$$E(X) = \sum_x x f(x) = \sum_{x=0}^{1} x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{1-x}$$

$$= 0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 = 0+\frac{1}{3} = \frac{1}{3}$$

$$E(2X+3) = 2E(X)+3 = 2\left(\frac{1}{3}\right)+3 = \frac{11}{3}$$

$$E(X^3) = \sum_{x=0}^{1} x^3\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{1-x}$$

$$= (0)^3\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right) + (1)^3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^0 = \frac{1}{3}$$

Some special mathematical expectation :-

1) The mean :-

If we let $u(x) = X$ then $E[u(X)] = E(X)$. This expectation, when it exists, is called the arithematic mean or simply, the mean of X.

The mean of X denoted by μ is defined to be;

$$\mu = E(X) = \begin{cases} \sum_{\text{all } x} x f_X(x) & \text{if } X \text{ is a discrete r.v.} \\ \int_{-\infty}^{\infty} x f_X(x)\, dx & \text{if } X \text{ is a continious r.v.} \end{cases}$$

2) The moments of a random variable:-

The moments (or raw moments) of a r.v. or a distribution are the expectation of the powers of the r.v. which has the given distribution.

Three kinds of moments are:-

i- The non-central moments or the moments about the origin:-

If X is a r.v., the mth moment of X, denoted by μ_m' is defined as:-

$\mu_m' = E(X^m)$

ii- The central moments :-

If X is a r.v., the mth central moment of X, about μ_X, denoted by

$\mu_m = E(X - \mu_X)^m$

and the m-th non-central moment of X is

defined as $E(X-a)^m$.

We see that the moment about the origin is a special case of non-central moment when $a=0$ and ~~also~~ the central moment is also t the special case of non central moment when $a=\mu_x$.

A final note on raw and central moments can be summerized in the following two points:-

a) If one of the two moments $E(X^m)$ and $E(X-\mu_x)^m$ exists, then the other moments of some order exists as well as moments of order less than m. That is, if $E(X^m)$ exists then $E(X-\mu_x)^m$ exists for all $n \leq m$ and Vice versa.

b) If one of the two moments $E(X^m)$ or $E(X-\mu_x)^m$ of a particular order does not exist, then the other moment of the same order does not exist and neither any other moment of order greater than m. That is, if $E(X^m)$ does not exist, then $E(X-\mu_x)^m$ does not exist for all $n \geq m$.

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If $E(X^2)$ exist, then $E(X)$ exist, also $E(X-\mu_x)^2$ exist and also $E(X-\mu_x)$. But if $E(X)$ does not exist, then neither $E(X-\mu_x)$ nor $E(X^2)$, $EX^3)$, ---- or $E(X-\mu_x)^2$, $E(X-\mu_x)^3$--- exist.

iii) The third Kind of moments are called factorial moments,

The mth factorial moment of a r.v. X (~~is~~ often discrete) is defined as:

$E[X(X-1)(X-2)---(X-m+1)]$ where m is a positive integer, number

3) The variance:-

Another special mathematical expectation is obtained by taking $u(x) = (X - \mu_X)^2$. The expected value of this function is called the variance of the r.v. X, and denoted by σ_X^2 or $Var(X)$. That is,

$$\sigma_X^2 = E(X - \mu_X)^2$$

$\sigma_X = \sqrt{\sigma_X^2}$ is called the standard deviation of the r.v. X, or of the distribution of the r.v. X.

Some theorems about the variance of a r.v. X are:-

theorem (1):-

$Var(X) \geq 0$.

proof:-

By definition

$\because Var(X) = E(X - \mu_X)^2 \geq 0$

$\therefore Var(X) \geq 0$,

theorem(2):-

$Var(C) = 0$, where C is a constant

proof:-

$Var(C) = E[C - E(C)]^2$, but since c is a constant, then $E(C) = C$ by the fact of expectation and therefore

$Var(C) = E(C - C)^2 = 0$

· **theorem (3):-**

$Var(CX) = C^2 Var(X)$

proof:-

$Var(CX) = E[CX - E(CX)]^2$

$= E[CX - C\mu_x]^2$

$= E[C^2(X - \mu_x)^2] = C^2 E(X - \mu_x)^2 = C^2 Var(X)$

theorem (4):-

$\sigma_x^2 = EX^2 - (E(X))^2 = EX^2 - \mu_x^2$

proof:-

$\sigma_x^2 = E(X - \mu_x)^2 = E(X^2 - 2X\mu_x + \mu_x^2)$

$= EX^2 - 2\mu_x EX + \mu_x^2$

$= EX^2 - 2\mu_x^2 + \mu_x^2 = EX^2 - \mu_x^2$

Ex (1):-

Let a r.v. x has the following p.d.f

$$f(x) = \begin{cases} \frac{4-x}{16} & -2 < x < 2 \\ 0 & o.w. \end{cases}$$

1- Find the third non-central moment.

2- Find the third central moment about 4.

3- Find the second central moment about the mean

sol:-

1) $\mu_3' = E(X^3) = \int_{-2}^{2} x^3 f(x)\,dx = \int_{-2}^{2} x^3 \frac{(4-x)}{16}\,dx$

$= \frac{1}{4}\int_{-2}^{2} x^3\,dx - \frac{1}{16}\int_{-2}^{2} x^4\,dx$

$= \frac{1}{16}(x^4 \Big|_{-2}^{2}) - \frac{1}{16(5)}(x^5 \Big|_{-2}^{2}) = \frac{1}{16}(16 - \frac{1}{16}) - \frac{1}{80}$

$\cdot(32+32) = 0.8$

2- $E(X-4)^3 = \int_{-2}^{2} (X-4)^3 \cdot \frac{(4-X)}{16}\, dx$

$= -\frac{1}{16} \int_{-2}^{2} (X-4)^4\, dx = -\frac{1}{16} \cdot \frac{(X-4)^5}{5} \Big|_{-2}^{2}$

$= -\frac{1}{16} \cdot \frac{(-2)^5}{5} + \frac{1}{16} \cdot \frac{(-6)^5}{5} = 0.4 - 97.2 = 96.8$

3- $\mu_2 = E(X-\mu_x)^2 = \sigma_x^2$

$EX = \int_{-2}^{2} X \cdot \frac{(4-X)}{16}\, dx = \frac{1}{16} \int_{-2}^{2} (4X - X^2)\, dx$

$= \frac{1}{16} \left[2X^2 \Big|_{-2}^{2} - \frac{X^3}{3} \Big|_{-2}^{2} \right] = \frac{1}{16} \left[8 - 8 - \left(\frac{8}{3} + \frac{8}{3}\right) \right]$

$= \frac{-1}{3}$

$\therefore E(X-\mu)^2 = E\left(X + \frac{1}{3}\right)^2$

$= \int_{-2}^{2} \left(X + \frac{1}{3}\right)^2 \cdot \frac{(4-X)}{16}\, dx$

$= \int_{-2}^{2} \left(X^2 + \frac{2}{3}X + \frac{1}{9}\right) \cdot \frac{(4-X)}{16}\, dx$

$= \frac{1}{16} \left[4 \int_{-2}^{2} X^2 dx + \frac{8}{3} \int_{-2}^{2} x\, dx + \frac{4}{9} \int_{-2}^{2} dx - \int_{-2}^{2} X^3 dx - \frac{2}{3} \right.$

$\left. \int_{-2}^{2} X^2 dx - \frac{1}{9} \int_{-2}^{2} x\, dx \right] = \frac{1}{16} \left[4 \left(\frac{X^3}{3} \Big|_{-2}^{2} \right) + \frac{4}{3} \left(X^2 \Big|_{-2}^{2} \right) \right.$

$\left. + \frac{2}{9} \left(X \Big|_{-2}^{2} \right) - \left(\frac{X^4}{4} \Big|_{-2}^{2} \right) - \frac{2}{9} \left(X^3 \Big|_{-2}^{2} \right) - \frac{1}{18} \left(X^2 \Big|_{-2}^{2} \right) \right] = 18.67$