Lecture (7)

- Probability generating function
- Joint distributions
- K- dimensional discrete and continuous r.v's.
- Joint p.d.f.

6) The probability generating function

The final special mathematical expectation is called the factorial moment generating function or brobability generating function which is defined as follows:-

The function $G_X(S) = E(S^X) = \sum_{n=0}^{\infty} S^n p(X=n)$ is called the factorial moment generating function (f.m.g.f) or the probability generating function (p.g.f) of the r.v. X or of the distribution of the r.v. X.

Properties:-

1- $G_X(1) = E(1^X) = E(1) = 1$

2- Factorial moments of the r.v. X when they exists, can be obtained from $G_X(S)$ as follows:-

$\because G_X(S) = E(S^X) = \sum_{n=0}^{\infty} S^n p(X=n) = p(X=0) + S\,p(X=1)$

$+ S^2 p(X=2) + S^3 p(X=3) + \cdots$

$= P_0 + SP_1 + S^2P_2 + S^3P_3 + \cdots$

$G_X'(S) = P_1 + 2SP_2 + 3S^2P_3 + \cdots = \frac{dG_X(S)}{dS}$

$\left.\frac{dG_X(S)}{dS}\right|_{S=0} = G_X'(0) = P_1$

$\Rightarrow G_X'(1) = \left.\frac{dG_X(S)}{dS}\right|_{S=1} = P_1 + 2P_2 + 3P_3 + \cdots$

$= \sum_{n=1}^{\infty} nP_n = \sum_{X=1}^{\infty} X\,P(X) = E(X)$

$\frac{d^2G_X(S)}{dS^2} = 2P_2 + 3.2SP_3 + 4.3S^2P_4 + \cdots$

$G_X''(0) = 2P_2$

$$\Rightarrow G_X''(1) = 2P_2 + 3.2P_3 + 4.3P_4 + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)P_n = E[X(X-1)]$$

$$G_X'''(s) = 3.2P_3 + 4.3.2SP_4 + 5.4.3S^2P_5 + \cdots$$

$$= 3!P_3 + 4!SP_4 + 5.4.3S^2P_5 + \cdots$$

$$G_X'''(0) = 3!P_3$$

$$G_X'''(1) = 3!P_3 + 4!P_4 + 5.4.3P_5 + \cdots$$

$$= \sum_{n=3}^{\infty} n(n-1)(n-2)P_n = E[(X)(X-1)(X-2)]$$

$$G_X^{(r)}(0) = \left.\frac{d^r G_X(s)}{ds^r}\right|_{s=0} = r!P_r \Rightarrow \boxed{P_r = p(X=r) = \frac{G_X^{(r)}(0)}{r!}}$$

$$G_X^{(r)}(1) = \left.\frac{d^r G_X(s)}{ds^r}\right|_{s=1} = E[X(X-1)(X-2)\cdots(X-r+1)]$$

the r-th factorial moment

Also the moment generating function $M_X(t)$ can be obtained from the p.g.f $G_X(s)$ by the following formula

$$M_X(t) = G_X(e^t)$$

or equivalently :-

$$G_X(s) = M_X(\ln(s))$$

Proof:-

$\because \mu_x(t) = E(e^{tx})$ and $G_x(s) = E(s^x)$

If $s = e^t \Rightarrow \ln(s) = t$, then

$G_x(e^t) = E[(e^t)^x] = E(e^{tx}) = \mu_x(t)$

$\mu_x(\ln s) = E[e^{x\ln s}] = E[e^{\ln(s^x)}] = E(s^x) = G_x(s)$.

Ex:-

Let $p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots$. Find the mean and variance of x and the rth factorial moment by using (p.g.f).

Sol:-

$G_x(s) = E(s^x) = \sum_{x=0}^{\infty} s^x \dfrac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \dfrac{(\lambda s)^x}{x!}$

$= e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}$

$\mu = E(x) = G_x'(1) = \dfrac{dG_x(s)}{ds}\Big|_{s=1} = \lambda e^{\lambda(s-1)}\Big|_{s=1} = \lambda$

$Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = E[X(X-1)] + E(X)$

$= G_x''(1) + G_x'(1)$

$\Rightarrow G_x''(1) = \dfrac{d^2 G_x(s)}{ds^2}\Big|_{s=1} = \lambda^2 e^{\lambda(s-1)}\Big|_{s=1} = \lambda^2$

$\Rightarrow E(X^2) = \lambda^2 + \lambda \quad \Rightarrow Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$

$G_x^{(r)}(1) = E[X(X-1)(X-2)\cdots(X-r+1)] = \lambda^r$

Chapter (2)

Joint and Conditional Distributions, Independence and Expectation:-

1) Joint distributions:-

If $X_1, X_2, \ldots, X_K$ are random variables, then $(X_1, X_2, \ldots, X_K)$ is called a K-dimensonal random variables.

Definition:- (1) K-dimensional r.v.

The K-dimensonal r.v. $(X_1, X_2, \ldots, X_K)$ is defined to be a K-dimensonal discrete r.v. if it can assume values only at a countable number of points $(X_1, X_2, \ldots, X_K) \in R^K$, the K-dimensonal real space.

If the result of an experiment is described by one ordered pair of numbers (X, Y), then we have two dimensional r.v.

Ex:-

اختيار Pick one student from the class, Let X be the height and Y be the weight of the student, then (X, Y) is a two-dimensonal r.v.

Ex:-

Toss two coins. let X be the result on the first coin and Y be the result on the second, then (X, Y) is a two-dimensional r.v.

Ex:-

Toss a pair of dice. Let X denote the result on the first die and Y be the result on the second then (X, Y) is a two dimensional r.v.

Definition :- K- dimensional discrete density function:-

If $(X_1, X_2, ---, X_K)$ is a K- dimensonal discrete r.v., then the joint discrete density function of $(X_1, X_2, ---, X_K)$, denoted by $f_{X_1, X_2, ---, X_K}(x_1, x_2, ---, x_K)$ which is defined by:-

$$f(x_1, x_2, ---, x_K) = p(X_1 = x_1, X_2 = x_2, ---, X_K = x_K)$$

for the particular case when (X, Y) is a two-dimensonal discrete r.v., it's distribution may be described as follows:-

(X, Y) : (X_1, Y_1), (X_2, Y_2), --- (X_n, Y_n) ---
$\qquad P_{11} \quad P_{22} \quad --- \quad P_{nn}$ ---

where

$$P_{jK} \geq 0 , \sum_j \sum_K P_{jK} = 1$$

and $P_{jK} = p(X = x_j, Y = y_K)$, $j = 1, 2, -- n$, $K = 1, -- n$

Sometimes, it is convenient to arrange the values and the probabilities of discrete random variables in the following ways:-

X \ Y	y_1	y_2	---	y_K ---
x_1	P_{11}	P_{12}	---	P_{1K}
x_2	P_{21}	P_{22}	---	P_{2K}
⋮	⋮	⋮		⋮
x_j	P_{j1}	P_{j2}	---	P_{jK}
⋮				

where $P_{jK} = p(X = x_j, Y = y_K) \geq 0$ and

$$\sum_{j,K} P_{jK} = 1$$

Ex:-

Let (X,Y): $(0,0)$, $(0,2)$, $(1,2)$, $(2,2)$

P_{ij} ; $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

Then (X,Y) is a two-dimonsional r.v. and it's distn. is defined, Also, we can present the distn. in a table as follows:-

X \ Y	0	2	
0	1/4	1/4	
1	0	1/4	
2	0	1/4	
			1

Definition:- K-dimensional continious r.v.

The K-dimensonal r.v. $(X_1, X_2, --, X_K)$ is defined to be a K-dimensonal continious r.v. if and only if there exists a function $f(x_1, x_2 --)$ such that

$P_{X_1, X_2, --X_K}(x_1, x_2 ..., x_K) \geq 0$ for all $(x_1, x_2 -- x_K)$ $\in R^d$ and

$$p(X_1 \leq x_1, X_2 \leq x_2, ..., X_K \leq x_K) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1, x_2 -- x_K)\, dx_1\, dx_2 ... dx_K$$

for all $(x_1, x_2, ---, x_K) \in R^K$. The function $f(x_1, x_2, ---, x_K)$ is called the joint p.d.f of the K-dimensonal r.v. $(X_1, X_2, --- X_K)$. It follows that the joint p.d.f $f(x_1, x_2, ---, x_K)$ has to satisfy that

$$\int_{-\infty}^{\infty} --- \int_{-\infty}^{\infty} f(x_1, x_2, ----, x_K)\, dx_1\, dx_2 ... dx_K = 1$$

For the particular case, the bivariate (2-dimensonal r.v. (X,Y) is a continious r.v. if and only if there exists a function $f(x,y) \geq 0$ for all $(x,y) \in R^2$ such that:-

$$p(X \leq x, Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\, du\, dv \quad ,(x,y) \in R^2$$

The joint p.d.f $f_{X,Y}(x,y)$ of the random variables X and Y must satisfy the following two properties:-

1) $f_{X,Y}(x,y) \geq 0$ for all $(x,y) \in R^2$

2) $\iint_{R^2} f(x,y)\, dx\, dy = 1$

Any function $f: R^2 \to R$ which satisfies the above two conditions or properties is a joint density function of a two dimensonal continuous r.v.

EX:-

Let $(X,Y) \sim f(x,y) = \begin{cases} 1 & 0 < x < 1, 0 < y < 1 \\ 0 & o.w. \end{cases}$

Then $f(x,y)$ is a joint density function for two dimensonal continuous r.v., that is

$$\iint_{R^2} f(x,y)\, dx\, dy = \int_0^1\int_0^1 dx\, dy = 1$$