Lecture (8)

- Examples on joint p.d.f.
- Joint c.d.f. with properties
- Marginal distributions with examples

Ex:-

Let $(X,Y) \sim f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{o.w.} \end{cases}$

then f is a joint density function for a two-dimensonal continuous r.v. and that

$$\iint_{R^2} f(x,y)\,dx\,dy = \int_0^1\int_0^1 (X+Y)\,dX\,dY$$

$$\int_0^1 \frac{X^2}{2} + YX \Big|_0^1 dY$$

$$= \int_0^1 \left(\frac{1}{2} + Y\right) dY = \frac{1}{2}Y + \frac{Y^2}{2}\Big|_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

Definition:- Joint comulative distribution function

Let (X,Y) be a jointly distributed r.v's., then a function $F_{X,Y}$ defined by:-

$$F_{X,Y}(x,y) = p(X \leq x, Y \leq y) \quad \forall (x,y) \in R^2$$

is called the joint comulative distn. function of the r.v. (X,Y).

Now if the continious r.v. (X,Y) has a known distn. function $F_{X,Y}(x,y)$, then its density function $f_{X,Y}(x,y)$ can be derived as follows:-

From the definition of the distn. function $F_{X,Y}(x,y)$ of this continuous r.v. (X,Y), we have

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is called the joint comulative $dist^{n.}$ function of the r.v. (X,Y).

Now if the continious r.v. (X,Y) has a known $dist^{n.}$ function $F_{X,Y}(x,y)$, then its density function $f_{X,Y}(x,y)$ can be derived as follows:-

From the definition of the $dist^{n.}$ function $F_{X,Y}(x,y)$ of this continuous r.v. (X,Y), we have

$$F(x,y) = P(X \leq x, Y \leq y) \quad , (x,y) \in R^2$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du$$

$$= \int_{-\infty}^{x}\left[\int_{-\infty}^{y} f(u,u)\,dv\right]du$$

If we take the derivative of $F(x,y)$ with respect to x first have:-

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial}{\partial x}\int_{-\infty}^{x}\left[\int_{-\infty}^{u} f(u,v)\,dv\right]du$$

$$= \int_{-\infty}^{y} f(x,v)\,dv$$

by the fundamental theorem of calculus.

Now, by taking the derivative with respect to the variable y, we get

$$\frac{\partial^2 F(x,y)}{\partial x\,\partial y} = \frac{\partial}{\partial y}\int_{-\infty}^{y} f(x,v)\,dv = f(x,y)$$

by the fundamental theorem of calculus.

Similar case holds for discrete r.v. (X,Y). Let $f(x,y)$ be the joint prob. function of (X,Y) then assuming that $(x_1,y_1),(x_2,y_2)\ldots(x_j,y_k)$ are the possible values of (X,Y). The c.d.f of (X,Y) $F(x,y)$ is given by

$$F(x,y) = p(X\leq x, Y\leq y) \quad ,(x,y)\in R^2$$

$$= \sum_{\substack{x_i\leq x\\ ,y_i\leq y}} p(X=x_i, Y=y_i)$$

Therefore

$$f(x_i, y_i) = F(x_i, y_i) - \lim_{0<h\to 0^-} F(x_i+h, y_i)$$

$$-\lim_{0<h<0^-} F(x_i, y_{i+h}) + \lim_{0<h\to \bar{0}} F(x_i-h, y_i-h)$$

$$= F(x_i, y_i) + F(x_i -, y_i) - F(x_i, y_{i-})$$

$$+ F(x_{i-}, y_{i-})$$

Let $(X,Y) \sim f(x,y) = \begin{cases} 1 & 0 \le X, Y < 1 \\ 0 & o.w. \end{cases}$

Find the following probabilities:-

$p(X \le \frac{1}{2}, Y \le \frac{1}{2})$, $p(\frac{1}{4} < X < \frac{1}{2}, \frac{1}{8} < Y < \frac{1}{2})$

and $p(X+Y < 1)$

Sol:-

$$p(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f(x,y)\,dx\,dy$$

$$= \int_0^{\frac{1}{2}} \left[\int_0^{\frac{1}{2}} dx\right] dy = \int_0^{\frac{1}{2}} [x]_0^{\frac{1}{2}}\, dy$$

$$= \int_0^{\frac{1}{2}} \frac{1}{2}\, dy = \frac{1}{2} y \Big|_0^{\frac{1}{2}} = \frac{1}{4}$$

$$p(\tfrac{1}{4} < X < \tfrac{1}{2}, \tfrac{1}{8} < Y < \tfrac{1}{4}) = \int_{\frac{1}{8}}^{\frac{1}{4}} \int_{\frac{1}{4}}^{\frac{1}{2}} dx\,dY$$

$$\int_{\frac{1}{8}}^{\frac{1}{4}} x \Big|_{\frac{1}{4}}^{\frac{1}{2}} dy = \int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{4}\, dy = \frac{1}{4} y \Big|_{\frac{1}{8}}^{\frac{1}{4}} = \frac{1}{32}$$

$$P(X+Y\le 1) = P(X\le 1-Y) = \int_0^1\int_0^{1-Y} dx\,dy$$

$$= \int_0^1 x\Big|_0^{1-Y} dY = \int_0^1 (1-Y)\,dY$$

$$= \left(Y-\frac{Y^2}{2}\right)\Big|_0^1 = 1-\frac{1}{2} = \frac{1}{2}$$

Marginal distributions:-

The marginal distⁿ. is the distribution of one r.v. out of the set of jointly distributed r.v's. If (X,Y) has a discrete distⁿ., then its distⁿ. can be presented in either values and probabilities or in a tabular form. Now, if we have

$$(X,Y) = (x_1,y_1),\ (x_1,y_2)\ \cdots\ (x_j,y_k),\ \cdots$$
$$\quad P_{11} \qquad P_{12} \qquad\qquad P_{ij}$$

where $P_{jk} = P(X=x_j,\ Y=y_k) \geq 0$ and $\sum_{j,k} P_{jk} = 1$

then the marginal distⁿ. of the discrete r.v. X is given by:

$$X:\ x_1 \quad x_2 \ \cdots\ x_n \cdots$$
$$\quad P_1 \quad P_2 \ \cdots\ P_n$$

where $P_j = \sum_k P_{jk} = P_{j1} + P_{j2} + \cdots + P_{jk} + \cdots$

and $\sum_j P_j = 1$

Similarly, the marginal distⁿ. of the discrete r.v. Y is given by:

$Y: \; y_1 \quad y_2 \; --- \; y_K \; ---$
$\quad\; P_{\cdot 1} \quad P_{\cdot 2} \quad\quad P_{\cdot K} \; ---$

where

$P_{\cdot K} = \sum_j P_{jK} = P_{1K} + P_{2K} + --- + P_{jK} + ---$

and $\sum_K P_{\cdot K} = 1$

Example

The marginal distⁿ· is given from the margins of the above table by:-

X \ Y	1	3	f(x)
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
2	0	$\frac{1}{3}$	$\frac{1}{3}$
f(y)	$\frac{1}{3}$	$\frac{2}{3}$	1

$X \; : \quad 1 \quad\quad 2$
$f(x) = \frac{2}{3} \quad \frac{1}{3}$

and the marginal distⁿ· of Y is

$Y \; : \quad 1 \quad\quad 3$
$f(Y) : \frac{1}{3} \quad \frac{2}{3}$

Ex:-

Let $(X,Y) : (1,1), (2,2), (3,3), (4,4), (5,5)$
$\quad\quad\quad\quad \frac{1}{5} \quad\; \frac{1}{5} \quad\; \frac{1}{5} \quad\; \frac{1}{5} \quad\; \frac{1}{5}$

Then $X : \; 1 \; , \; 2 \; , \; 3 \; , \; 4 \; , \; 5$
$\quad\quad\quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5}$

represents the marginal distⁿ· of X.

Ex:-
Let (X, Y): $(0,0)$, $(0,1)$, $(0,2)$, $(1,2)$
$f(x,y)$: $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{4}$

Then the marginal distⁿ of X and of Y are given by:

X:	0	1	and	Y:	0, 1, 2	
$f(x)$:	$\frac{3}{4}$	$\frac{1}{4}$		$f(y)$:	$\frac{1}{4}$ $\frac{1}{4}$ $\frac{2}{4}$	

If we let the continuous r.v. (X, Y) whose p.d.f $f(x,y)$ then

$$F(x,y) = P(X \leq x, Y \leq y) \qquad x, y \in R^2$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\, du\, dv$$

Then the r.v. X has a c.d.F $F(x)$ which is given by:-

$$F(x) = P(X \leq x) = P(X \leq x, Y \leq \infty)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(u,y)\, du\, dy = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,y)\, dy\, du$$

Then X has a p.d.f equals to:

$$f(x) = \frac{d}{dx} F(x)$$

$$= \frac{d}{dx} \int_{-\infty}^{x} \int_{-\infty}^{\infty} f(u,y)\, dy\, du$$

$$= \int_{-\infty}^{\infty} f(x,y)\, dy$$

Similarly the marginal p.d.f of the r.v. Y
given by :-

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Ex:- (imp.)

Let $(X, Y) \sim f(x, y) = \begin{cases} 1 & 0 < X < 1,\ 0 < Y < \\ 0 & o.w. \end{cases}$

Then the marginal p.d.f of X is

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \int_{0}^{1} dy = \begin{cases} 1 & 0 < X < 1 \\ 0 & o.w. \end{cases}$$

Also the marginal p.d.f of Y is

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_{0}^{1} dx = \begin{cases} 1 & 0 < Y < 1 \\ 0 & o.w. \end{cases}$$

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