Lecture (9)

- Conditional distributions
- Independence of random variables
- Remarks and examples

3) Conditional distributions

We will consider the distⁿ of one random variable conditioned by the occurrence of another one. That is we will study the distⁿ of X given $Y=y$ or the distⁿ of Y given $X=x$, where (X,Y) is a two dimensional r.v. with a specified joint probability density function $f(x,y)$.

If we have the joint p.d.f $f(x,y)$ and the marginal density functions of X and Y, which are $f(x)$ and $f(y)$, respectively, then for $f(x)>0$ the conditional distⁿ of Y given $X=x$ can be defined as follows:-

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad \text{or} \quad f(y|x) = \frac{f(x,y)}{f(x)}$$

Similarly, for $f(y)>0$, the conditional distⁿ of X

given $Y=y$ is given by:-

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$f(x|y)$ must satisfy the conditions of being a probability function.

Ex:-

Let $(X,Y) = (0,0),\ (0,1),\ (1,1)$

$\qquad\qquad\qquad\ \frac{1}{3} \qquad\ \frac{1}{3} \qquad\ \frac{1}{3}$

then $Y = 0 \qquad 1$

$\quad f(y): \frac{1}{3} \qquad \frac{2}{3}$

And $X|(Y=0): 0$, which is called the degenerate r.v. at $x=0$

$f(X|Y=0) = 1 \qquad \Rightarrow f(x|y=0) = \frac{f(0,0)}{f(y=0)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$

$X|Y=1 : 0$ or 1

$f(X|Y=1): \frac{1}{2} \quad , \frac{1}{2} \qquad f(0|y=1) = \frac{f(0,1)}{f(y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

$$f(1|y=1) = \frac{f(1,1)}{f(y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Example:-

Let $(X_1,X_2) \sim f(x_1,x_2) = \begin{cases} 2 & 0<x_1<x_2<1 \\ 0 & o.w. \end{cases}$

Find

a) $f(x_1)$ and $f(x_2)$

b) $f(x_1|x_2)$ and $f(x_2|x_1)$

c) $P(0<X_1<\frac{1}{2})$, $P(0<X_1<\frac{1}{2} | X_2=x_2)$ and $P(0<X_1<\frac{1}{2} | X_2=\frac{3}{4})$

Sol:-

$$f(x_1) = \int_{x_2=x_1}^{x_2=1} 2\,dx_2 = 2x_2 \Big|_{x_1}^{1} = 2(1-x_1)$$

$$\therefore f(x_1) = \begin{cases} 2(1-x_1) & 0<x_1<1 \\ 0 & o.w \end{cases}$$

Similarly

$$f(x_2) = \int_{x_1=0}^{x_1=x_2} 2\,dx_1 = 2x_2$$

$$\therefore f(x_2) = \begin{cases} 2x_2 & 0<x_2<1 \\ 0 & o.w \end{cases}$$

b-) $f(x_1/x_2) = \dfrac{f(x_1,x_2)}{f(x_2)} = \dfrac{2}{2x_2} = \dfrac{1}{x_2}$

$$\therefore f(x_1/x_2) = \begin{cases} \dfrac{1}{x_2} & 0<x_1<x_2 , \; 0<x_2<1 \\ 0 & o.w. \end{cases}$$

and similarly

$f(x_2/x_1) = \dfrac{f(x_2,x_1)}{f(x_1)} = \dfrac{2}{2(1-x_1)} = \dfrac{1}{1-x_1}$

$$\therefore f(x_2/x_1) = \begin{cases} \dfrac{1}{1-x_1} & x_1<x_2<1 , \; 0<x_1<1 \\ 0 & o.w. \end{cases}$$

c) $p(0<x_1<\frac{1}{2}) = \int_0^{\frac{1}{2}} f(x_1)\,dx_1 = 2\int_0^{\frac{1}{2}} (1-x_1)\,dx_1$

$= 2x_1 - 2\dfrac{x_1^2}{2} \Big|_0^{1/2} = \dfrac{3}{4}$

$p(\frac{1}{2}<x_1<\frac{1}{2} \mid X_2 = x_2) = \int_0^{\frac{1}{2}} = f(x_1 \mid x_2) = dx_1$

$$= \int_0^{\frac{1}{2}} \frac{1}{X_2}\, dX_1 = \frac{X_1}{X_2}\Big|_0^{\frac{1}{2}} = \frac{\frac{1}{2}}{X_2} = \frac{1}{2X_2}$$

$$p(0 < X_1 < \tfrac{1}{2} \mid X_2 = \tfrac{3}{4}) = \int_0^{\frac{1}{2}} f(X_1 \mid X_2 = \tfrac{3}{4})\, dX_1$$

$$= \int_0^{\frac{1}{2}} \frac{4}{3}\, dX_1 = \frac{4}{3} X_1\Big|_0^{\frac{1}{2}} = \frac{4}{3}\left(\frac{1}{2}\right) = \frac{2}{3}$$

او التعويض عن X_2 بـ $\frac{3}{4}$ في الاحتمالية السابقة

$$\therefore\ p(0 < X_1 < \tfrac{1}{2} \mid X_2 = x_2) = \frac{1}{2x_2}$$

if $X_2 = \frac{3}{4}$

$$\Rightarrow p(0 < X_1 < \tfrac{1}{2} \mid X_2 = \tfrac{3}{4}) = \frac{1}{2(\frac{3}{4})} = \frac{2}{3}$$

Independence of random variables:-

If A and B are two events, then A and B are independent iff $p(A \cap B) = p(A)\,p(B)$

Therefore, the definition for the independence of two random variables, X and Y can be given similarly as follows:-

Definition (1):-

Let X and Y be two r.v's. Then X and Y are indep. iff $\forall A = (a,b]$, $B = (c,d]$, $a \le b$ and $c \le d$ we have:-

$$p(X \in A, Y \in B) = p(X \in A)\, p(Y \in B)$$

theorem:-

~~Let~~ X and Y are independent iff $\forall (x,y) \in R^2$

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$$

theorem:-

Let X and Y be two r.v's. then X and Y are independent iff:-

$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ for all $(x,y) \in R^2$

where $f(x,y)$ is the joint p.d.f. of X and Y, and $f(x)$ and $f(y)$ are the marginal p.d.f's of X and Y, respectively

Example:-

Let $(X,Y) = \underset{\frac{1}{3}}{(0,0)},\ \underset{\frac{1}{3}}{(1,0)},\ \underset{\frac{1}{3}}{(1,2)}$

a) Find the marginal distributions and conditional distributions.

b) Are X and Y independent?

c) Find $p(X=0)$, $P(X=1)$, $p(X\leq 0)$, $p(X=1, Y=0)$, $P(-1\leq Y\leq 2)$, $P(-\frac{1}{2}< Y< 1)$, $p(\frac{1}{2}< X< \frac{3}{2}, 1\leq Y\leq 2)$

sol:-

a) X : 0 , 1

$f(x)$: $\frac{1}{3}$, $\frac{2}{3}$

Y : 0 , 2

$f(y)$: $\frac{2}{3}$, $\frac{1}{3}$

$$p(X=x\,|\,Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

For this case

$$P(X=0\,|\,Y=0) = \frac{P(0,0)}{P(Y=0)} = \frac{\frac{1}{3}}{2/3} = \frac{1}{2}$$

$$p(X=1\,|\,Y=0) = \frac{P(1,0)}{P(Y=0)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Therefore:

$X|Y=0$: 0 , 1

$f(X|Y=0)$: $\frac{1}{2}$ $\frac{1}{2}$

Similarly

$$p(X=1/Y=2) = \frac{p(1,2)}{p(Y=2)} = \frac{1/3}{1/3} = 1$$

For the conditional probabilities of Y given X, we have:-

$$p(Y=0|X=0) = \frac{p(X=0,Y=0)}{p(X=0)} = \frac{1/3}{1/3} = 1$$

Similarly

$Y|X=1$: 0 , 2

$p(Y|X=1)$: $\frac{1}{2}$ $\frac{1}{2}$

b) X and Y are not independent since $f(x,y) \neq f(x)f(y)$

c) 1) $p(X=0) = \frac{1}{3}$

2) $p(X=1) = \frac{2}{3}$

3) $p(X\leq 0) = p(X=0) = \frac{1}{3}$

4) $p(X=1, Y=0) = \frac{1}{3}$

5) $p(-1\leq Y\leq 2) = p(Y=0) + p(Y=2) = \frac{2}{3} + \frac{1}{3} = 1$

6) $p(-\frac{1}{2} < Y < 1) = p(Y=0) = \frac{2}{3}$

7) $p(\frac{1}{2} < X < \frac{3}{2}, 1\leq Y\leq 2) = p(X=1, Y=1) + p(X=1, Y=2)$

$= 0 + \frac{1}{3} = \frac{1}{3}$

Ex(2):-

If $f(x_1, x_2) = e^{-(x_1+x_2)}$ $\quad 0<x_1<\infty, \; 0<x_2<\infty$

1- Show that $f(x_1, x_2)$ is a joint p.d.f.

2- Find $P(0<X_1<2, \; 1<X_2<3)$

3- The joint c.d.f of X_1 and X_2

sol:-

1) $$\int_{X_1}\int_{X_2} f(x_1, x_2) = \int_0^\infty \int_0^\infty e^{-(x_1+x_2)}\, dx_1\, dx_2$$

$$= \int_0^\infty e^{-x_2} \left. \frac{e^{-x_1}}{-1} \right|_0^\infty dx_2$$

$$= \int_0^\infty e^{-x_2}\, dx_2 = 1$$

2- $$P(0<X_1<2, \; 1<X_2<3) = \int_1^3 \int_0^2 (e^{-(x_1+x_2)})\, dx_1\, dx_2$$

$$= \int_1^3 e^{-x_2} \left[-e^{-x_1}\right]_0^2 dx_2$$

$$= (1-e^{-2}) \int_1^3 e^{-x_2}\, dx_2 = (1-e^{-2})(e^{-1}-e^{-3})$$

$$= e^{-1} - e^{-3} - e^{-3} + e^{-5} = e^{-1} - 2e^{-3} + e^{-5}$$

$$= e^{-5}(1 - 2e^{2} + e^{4}) = \cancel{e^{-5}}$$

3) $$F(x_1, x_2) = \int_0^{x_1}\int_0^{x_2} e^{-(x_1+x_2)}\, dx_2\, dx_1$$

$$= \int_0^{x_1} e^{-x_1} \left[-e^{-x_2}\right]_0^{x_2} dx_1 = (1-e^{-x_2})(1-e^{-x_1})$$

Ex (1)

If you have the following joint p.d.f

$$f(x_1, x_2) = \begin{cases} \dfrac{2x_1 - x_2}{21} & x_1 = 1,2,3 \quad , x_2 = 0,1 \\ 0 & \text{o.w.} \end{cases}$$

1- Show that $f(x_1, x_2)$ is a joint p.d.f.
2- Find $p(X_1 = 2, X_2 = 1)$
3- Find the marginal p.d.f of x_1 and x_2.
4- Are X_1 and X_2 independent?

Sol:-

1) $$\sum_{x_1}\sum_{x_2} f(x_1, x_2) = \frac{1}{21}\left[\sum_{x_1=1}^{3}\sum_{x_2=0}^{1}(2x_1 - x_2)\right]$$

$$= \frac{1}{21}\sum_{x_1=1}^{3}\left[2x_1\sum_{x_2=0}^{1}(1) - 1\right]$$

$$= \frac{1}{21}\left(\sum_{x_1=1}^{3}(4x_1) - 3\right) = \frac{24-3}{21} = 1$$

2) $$p(x_1 = 2, x_2 = 1) = \frac{4-1}{21} = \frac{1}{7}$$

3) $$f(x_1) = \sum_{x_2=0}^{1}\left(\frac{2x_1 - x_2}{21}\right) = \frac{4x_1 - 1}{21} \qquad x_1 = 1,2,3$$

$$f(x_2) = \sum_{x_1=1}^{3}\left(\frac{2x_1 - x_2}{21}\right) = \frac{12 - 3x_2}{21} \qquad x_2 = 0,1$$

$$= \frac{4 - x_2}{7} \qquad x_2 = 0,1$$

$$f(x_1)f(x_2) = \frac{(4x_1 - 1)(4 - x_2)}{21(7)} = \frac{16x_1 - 4 + 4x_1x_2 + x_2}{7(21)}$$

$$= \frac{4x_1x_2 + 16x_1 + x_2 - 4}{7(21)} \neq f(x_1, x_2)$$

$\Rightarrow$ X_1 and X_2 are dependent variables