Lecture (10)

- Examples on conditional dist.
- Joint expectations
- Covariance and Correlation

Ex(7):-

Suppose that the p.d.f of X_1 and X_2 is specified as follows:-

(x_1, x_2) :	(0,1)	(0,2)	(0,3)	(1,1)	(1,2)	(1,3)
$f(x_1, x_2)$:	0.1	0.2	0.3	0.2	0.1	0.1

Find

1- The marginal distribution for each variable.

2- $Pr(X_1=0 / X_2=x_2)$, $P(X_1=1/X_2=x_2)$

$P(X_1=0)$, $P(X_1=1)$, $P(X_1 \leq 0)$, $P(X_2=0)$

$P(X_2 \geq 1)$

Sol:-

$X_1 \backslash x_2$	1	2	3	$f(x_1)$
0	0.1	0.2	0.3	0.6 ⇒ $f(x_1)$
1	0.2	0.1	0.1	0.4 ⇒ $f(x_2)$
$f(x_2)$	0.3	0.3	0.4	0.1

$$f(x_1) = \sum_{x_2=\phi}^{3} f(x_1, x_2) = \begin{cases} 0.6 & \text{for } x_1=0 \\ 0.4 & \text{for } x_1=1 \end{cases}$$

$$f(x_2) = \sum_{x_1=0}^{1} f(x_1, x_2) = \begin{cases} 0.3 & x_2=1 \\ 0.3 & x_2=2 \\ 0.4 & x_2=3 \end{cases}$$

$$f(X_1=0 | x_2=x_2) = \frac{f(x_1=0, X_2=x_2)}{f(x_2)} = \begin{cases} \frac{0.1}{0.3} = \frac{1}{3} & \text{if } X_2=1 \\ \frac{0.2}{0.3} = \frac{2}{3} & \text{if } X_2=2 \\ \frac{0.3}{0.4} = \frac{3}{4} & \text{if } X_2=3 \end{cases}$$

$$P(X_1=1 / X_2=x_2) = \begin{cases} 0.2/0.3 = \frac{2}{3} & \text{if } X_2=1 \\ 0.1/0.3 = \frac{1}{3} & \text{if } X_2=2 \\ 0.1/0.4 = \frac{1}{4} & \text{if } X_2=3 \end{cases}$$

$p(X_1=0)=0.6$ $p(X_1=1)=0.4$ $p(X_1<0)=0$

$p(X_2=0)=0$

Expectations:-

Let $(X_1, X_2, \dots, X_K)$ be a k-dimensional r.v. with joint probability function $f(x_1, x_2, \dots, x_K)$. The expected value of a function $g(\cdot,\dots)$ of the k-dimensional r.v. denoted by;

$E(g(x_1, x_2, \dots, x_K))$ is defined as:-

$$E[g(x_1, x_2, \dots, x_K)] = \begin{cases} \sum\limits_{x_1,\dots,x_K} g(x_1, x_2 \dots x_K) f(x_1, x_2 \dots x_K) & \text{if the r.v. } (X_1, X_2 \dots X_K) \text{ are discrete.} \\ \int\limits_{x_1}\int\limits_{x_2}\dots\int\limits_{x_K} g(x_1, x_2, \dots, x_K) f(x_1, x_2, \dots, x_K)\, dx_1 dx_2 \dots dx_K & \text{if } X_1, X_2 \dots X_K \text{ are continuous r.vs.} \end{cases}$$

Therefore for a two-dimensional r.v. (X,Y), the expected value of the function $g(x,y)$ is given by

$$E(g(x,y)) = \begin{cases} \sum\limits_{x_j}\sum\limits_{y_K} g(x_j, y_K) f(x_j, y_K) & \text{if } (X,Y) \text{ is discrete r.v.} \\ \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy & \text{if } (X,Y) \text{ is continuous r.v.} \end{cases}$$

theorem:-

a) If $g(x_1, x_2 \dots x_K) = x_i$ then

$E[g(x_1, x_2, \dots, x_K)] = E(x_i) = \mu_i$

b) If $g(x_1, x_2, \dots, x_K) = (x_i - \mu_i)^2$ then

$E[g(x_1, x_2, \dots, x_K)] = E(x_i - \mu_i)^2 = Var(x_i) = \sigma_i^2$

Definition:-

Let X and Y be any two r.v's, defined on the same sample space. The covariance of X and Y denoted by $Cov(X,Y)$ or $\sigma_{X,Y}$ is defined as:-

$Cov(X,Y) = E(X-\mu_x)(Y-\mu_y)$

theorem:-

$Cov(X,Y) = E(XY) - E(X)E(Y)$

proof:-

$Cov(X,Y) = E(X-\mu_x)(Y-\mu_Y)$

$= E[XY - \mu_x Y - \mu_y X + \mu_x \mu_y]$

$= E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x \mu_y$

$= E(XY) - \mu_x \mu_y$

$= E(XY) - E(X)E(Y)$

Definition:-

The correlation coefficient, denoted by $\rho_{x,y}$ or $\rho_{(X,Y)}$ of the r.v's. X and Y is defined as:

$$\rho_{x,y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

Ex:-

Let (X,Y) : $(0,0)$, $(1,2)$, $(1,3)$ with

$f(X,Y)$: $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

Find

a) $E(X)$, b) $Var(X)$ c) $Cov(X,Y)$ d) $\rho_{x,y}$

e) $E(X+Y)$ f) $E(X^2 + \frac{1}{2}Y)$

Sol:-

Y \ X	0	1	f(y)
0	1/3	0	$\frac{1}{3}$
2	0	$\frac{1}{3}$	$\frac{1}{3}$
3	0	$\frac{1}{3}$	$\frac{1}{3}$
f(X)	$\frac{1}{3}$	$\frac{2}{3}$	1

a) $X: \quad 0 \quad , \quad 1$

$f(x): \quad \frac{1}{3} \quad , \quad \frac{2}{3}$

$$E(X) = \sum_{X=0}^{1} X f(x) = 0\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = \frac{2}{3}$$

b) $$E(X^2) = \sum_{X=0}^{1} X^2 f(x) = 0\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) = \frac{2}{3}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$$

c) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$Y: \quad 0 \quad , \quad 2 \quad , \quad 3$

$f(y): \quad \frac{1}{3} \quad , \quad \frac{1}{3} \quad , \quad \frac{1}{3}$

$$E(Y) = \sum_{K=1}^{3} Y_K f(y_K) = 0\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) = \frac{5}{3}$$

$$E(XY) = \sum X_i Y_j f(X_i, Y_j) = 0(0)\left(\frac{1}{3}\right) + (1)(2)\left(\frac{1}{3}\right) + (1)(3) \cdot \left(\frac{1}{3}\right) = \frac{5}{3}$$

Therefore $Cov(X,Y) = \frac{5}{3} - \left(\frac{2}{3}\right)\left(\frac{5}{3}\right) = \frac{5}{3} - \frac{10}{9} = \frac{5}{9}$

d) $$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

$$E(Y^2) = \sum_{K=1}^{3} Y_K^2 f(y_K) = (0)^2\left(\frac{1}{3}\right) + (2)^2\left(\frac{1}{3}\right) + (3)^2\left(\frac{1}{3}\right)$$

$$= \frac{4}{3} + \frac{9}{3} = \frac{13}{3}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{13}{3} - \left(\frac{5}{3}\right)^2 = \frac{14}{9}$$

$$\therefore \rho_{X,Y} = \frac{\frac{5}{9}}{\sqrt{\left(\frac{2}{9}\right)\left(\frac{14}{9}\right)}} = \frac{5}{\sqrt{28}} = 0.94$$

e) $$E(X+Y) = E(X) + E(Y) = \frac{2}{3} + \frac{5}{3} = \frac{7}{3}$$

k) $E(X^2 + \frac{1}{2}Y) = E(X^2) + \frac{1}{2}E(Y) = \frac{2}{3} + \frac{1}{2}(\frac{5}{3}) = \frac{3}{2}$

Ex:-

Let $(X,Y) \sim f(x,y) = \begin{cases} 1 & 0 \le x \le 1 \ , \ 0 \le y \le 1 \\ 0 & o.w. \end{cases}$

Find

a) $E(X)$, b) $E(X^2)$, c) $Var(X)$, d) $E(Y)$, e) $E(Y^2)$

f) $Var(Y)$, g) $E(XY)$, h) $Cov(XY)$, i) $\rho_{X,Y}$

j) $E(X+Y)$ and k) $E(X^2 + \frac{1}{2}Y)$

Sol:-

a) $E(X)$, $g(X,Y) = X$ then

$$E(X) = \int_0^1\int_0^1 x \, dx dy$$

$$= \int_0^1 \frac{x^2}{2}\Big|_0^1 dy = \int_0^1 \frac{1}{2} dy = \frac{1}{2}y\Big|_0^1 = \frac{1}{2}$$

b) $E(X^2) = \int_0^1\int_0^1 x^2 dx dy = \int_0^1 \frac{x^3}{3}\Big|_0^1 dy = \frac{1}{3}y\Big|_0^1 = \frac{1}{3}$

c) $Var(X) = E(X^2) - (E(X))^2 = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$

d) $E(Y) = \int_0^1\int_0^1 y \, dx dy = \int_0^1 yx\Big|_0^1 dy = \int_0^1 y dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$

e) $E(Y^2) = \int_0^1\int_0^1 y^2 dx dy = \int_0^1 y^2 x\Big|_0^1 dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3}$

f) $var(Y) = E(Y^2) - (E(Y))^2 = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$

g) $E(XY) = \int_0^1\int_0^1 xy \, dx dy = \int_0^1 y\frac{x^2}{2}\Big|_0^1 dy = \frac{y^2}{4}\Big|_0^1 = \frac{1}{4}$

h) $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{1}{2}(\frac{1}{2}) = 0$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{0}{\sqrt{}} = 0$$

$$E(X+Y) = E(X) + E(Y) = \frac{1}{2} + \frac{1}{2} = 1$$

or $$E(X+Y) = \int_0^1\int_0^1 (X+Y)\,dx\,dY = \int_0^1 \left(\frac{X^2}{2} + YX\right)\Big|_0^1 dY$$

$$= \int_0^1 \left(\frac{1}{2} + Y\right) dY = \frac{1}{2}Y + \frac{Y^2}{2}\Big|_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

$$E\left(X^2 + \frac{1}{2}Y\right) = E(X^2) + \frac{1}{2}E(Y) = \frac{1}{3} + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{7}{12}$$