احصاء رياضي (٢١٠)

Lecture (1)

- Random variables
- Probability mass and density functions
- Cumulative distribution function
- Examples.

Chapter (1)

Basic Review

Random variable :-

Given a random experiment with it's sample space. A random variable (r.v.) denoted by X, is a function which assigns to each outcome $\omega \in S$ one and only one real number $X(\omega)$.

Remark:-

A r.v. will be denoted by a capital letter $X, Y, Z, \ldots$ or $X_1, X_2, \ldots$. The real numbers $X(\omega)$ will be denoted by x, ie $X(\omega) = x$.

The space of a r.v. X is the set of real numbers $\mathcal{A}$ such that $\mathcal{A} = \{x : X(\omega) = x, \omega \in S\}$

Ex:-

If an unbiased coin tossed twice, then the sample space is:-

$S = \{\omega$, where ω is (T, T) or (T, H) or (H, T) or $(H, H)\}$

Then let the interest be in the number of heads observed:

$X(\omega) = 0$ if ω is (T, T) (no heads)

$X(\omega) = 1$ if ω is either (T, H) or (H, T) (one head)

$X(\omega) = 2$ if ω is (H, H) (two heads).

The sample space of the random variable X is $\mathcal{A} = \{x ; x = 0, 1, 2\}$

Then $p(X=0) = \frac{1}{4}$, $p(X=1) = \frac{1}{2}$ and $p(X=2) = \frac{1}{4}$

Therefore, we can have the following tabular form:-

X	0	1	2
$p(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Discrete random variable :-

A r.v. is called discrete if its values are finite or countable in number, that is a r.v. is discrete if its range is a discrete set of real numbers.

The probability mass function:-

If X is discrete r.v., then a non-negative real-valued function f, defined on R by:-

$f(x) = p(X=x)$, $x \in R$

is called the probability mass function (p.m.f) of the r.v. X.

Often p.m.f is called the discrete density function.

Note that, any function f satisfies the following :-

i) $f(x) \geq 0$ for all $x \in R$

ii) $\sum_{\text{all } x \in R} f(x) = 1$

is a p.m.f of some r.v.

Continious random variable:-

A r.v. X is said to be an absoluately continious, or simply continious, if there exists a function f such that

$$p(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \qquad \text{for all } x \in R$$

Probability density function:-

Let X be a continious random variable. A non-negative real-valued function f such that

$$p(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{for all } x \in R$$

is called the probability density functiom (p.d.f) of the r.v. X.

Note that, any function f satisfying the following:-

i) $f(x) \geq 0$ for all $x \in R$.

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

is a p.d.f of some r.v.

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Ex (1):-

Let X be a r.v. of the discrete type with space $\mathcal{A} = \{x;\ x = 1, 2, 3, \ldots\}$ and let the pmf. of X be:-

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^{x} & x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

If $A = \{x\ ;\ x = 1, 3, 5, 7, \ldots\}$. Find $p(x \in A)$

Sol:-

Using the definition of p.m.f, we have

$$p(X \in A) = \sum_{X \in A} p(X = x) = p(X=1) + p(X=3) + p(X=5) + p(X=7) + \ldots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \ldots$$

$$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \ldots\right] = \frac{1}{2}\left[\frac{1}{1 - \frac{1}{4}}\right] = \frac{2}{3}$$

Therefore $p(x \in A) = \frac{2}{3}$

Ex (2):-

Let X be a r.v. of the continious type defined on the space $\mathcal{A} = \{x : 0 < x < \infty\}$, and let the p.d.f of x be :-

$$f(x) = \begin{cases} e^{-x} & x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

If $A = \{x ; 1 < x < 2\}$. Find $p(x \in A)$

Sol:-

$$p(x \in A) = p(1 < x < 2) = \int_1^2 e^{-x} dx = -e^{-x} \Big|_1^2 = -(e^{-2} - e^{-1})$$

$$= 0.23$$

Ex(3):-

Let X be a r.v. with a p.d.f given by:-

$$f(x) = \begin{cases} cx(2-x) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of the constant c such that f is a p.d.f.

Sol:-

From the property of a p.d.f., we must have;

$$\int_0^2 f(x) dx = 1$$

$$\int_0^2 cx(2-x) dx = 1 \Rightarrow 2c \int_0^2 x\, dx - c \int_0^2 x^2 dx = 1$$

$$\Rightarrow 2c \left[\frac{x^2}{2} \Big|_0^2 \right] - c \left[\frac{x^3}{3} \Big|_0^2 \right] = 1$$

$$2c\left(\frac{4}{2}\right) - c\left(\frac{8}{2}\right) = 1 \Rightarrow c = \frac{3}{4}$$

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Ex (4):-

Let a function f be defined as:-

$$f(x)=\begin{cases}\frac{k}{x} & x=1,2,3\\ 0 & \text{otherwise}\end{cases}$$

discrete

Find the value of k which makes f a p.d.f.

Sol:-

$\because \sum_{-\infty}^{\infty} f(x) = 1$ (p.m.f)

$\therefore \sum_{x=1}^{3} \frac{k}{x} = 1 \Rightarrow k + \frac{k}{2} + \frac{k}{3} = 1 \Rightarrow k(1+\frac{1}{2}+\frac{1}{3}) = 1$

$k(\frac{6+3+2}{6}) \Rightarrow k = \frac{6}{11}$

Ex (5):-

Find the value of k which makes f given by:

$$f(x)=\begin{cases}e^{-kx} & x>0\\ 0 & \text{o.w}\end{cases}$$

a p.d.f.

Sol:-

$\because \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} e^{-kx}\,dx = 1 \Rightarrow -\frac{1}{k}\left[e^{-kx}\Big|_{0}^{\infty}\right] = 1$

$\Rightarrow -\frac{1}{k}\left[e^{-\infty} - e^{0}\right] = 1 \Rightarrow -\frac{1}{k}[0-1] = 1 \Rightarrow k = 1$

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Ex (6):-

Find the value of k in the following tables which makes $p(x)$ given by the tables, is p.m.f.

①

x	0	1	2	3
$p(x)$	k	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$

②

x	1	2	3	4
$p(x)$	$\frac{1}{2}$	$\frac{k}{12}$	$\frac{k}{6}$	$\frac{k}{3}$

Sol:-

① $\sum_{x=0}^{3} p(x) = 1 \Rightarrow k + \frac{1}{6} + \frac{1}{3} + \frac{1}{12} = 1 \Rightarrow k + \frac{2+4+1}{12} = 1$

$\Rightarrow k + \frac{7}{12} = 1 \Rightarrow k = \frac{5}{12}$

$\therefore$ the prob. dist$^{n.}$ table is:-

x	0	1	2	3
$p(x)$	$\frac{5}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$

② $\sum_{x=1}^{4} p(x) = 1 \Rightarrow \frac{1}{2} + \frac{k}{12} + \frac{k}{6} + \frac{k}{3} = 1$

$\Rightarrow \frac{1}{2} + \frac{k+2k+4k}{12} = 1 \Rightarrow \frac{1}{2} + \frac{7k}{12} = 1$

$\Rightarrow k = \frac{1}{2}\left(\frac{12}{7}\right) = \frac{6}{7}$

$\therefore$ The prob. dist$^{n.}$ table is

x	1	2	3	4
$p(x)$	$\frac{1}{2}$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{2}{7}$

The Comulative Distribution Function:-

The comulative distribution function (c.d.f) of a r.v. X denoted by F_X, is defined by:-

$$F_X(x) = p(X \leq x) \quad , -\infty < x < \infty$$

Thus, for a discrete r.v., X
$F_X(x) = \sum_{u \leq x} f(u)$, where $f(x)$ is the p.m.f of X,

and for a continious r.v. X
$F_X(x) = \int_{-\infty}^{x} f(u)\, du$, where $f(x)$ is the p.d.f of X,

conversely, if the camulative distribution function (c.d.f) of a r.v. is Known, it can be used to find the probability function of that random variable.
For a discrete r.v., X with values $x_1, x_2, \dots, x_j$, we have;
$f(x_j) = p(X = x_j) = F(x_j) - F(x_j -)$, where
$F(x_j -) = \lim_{h \to 0} F(x_j - 1)$ $\quad f(x_j) = F(x_j) - F(x_{j-1})$ يعني

For a continious r.v., we need only to take the derivative of $F(x)$ with respect to x to find the p.d.f of x, that is:
$f(x) = F'(x) = \dfrac{dF(x)}{dx}$ for all x where F is differentiable.

Ex:-

Find the value of K which makes the following functions p.d.f.

① $f(x) = \begin{cases} K(1-x) & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

② $f(x) = \begin{cases} Ke^{-3x} & x>0 \\ 0 & \text{o.w} \end{cases}$

③ $f(x) = \begin{cases} Kxe^{-x^2/2} & x>0 \\ 0 & \text{o.w.} \end{cases}$

④ $f(x) = \begin{cases} Kx & x=0,1,2,3,4 \\ 0 & \text{o.w.} \end{cases}$

⑤ $f(x) = \begin{cases} Kx^2 & x=0,1,2 \\ 0 & \text{o.w.} \end{cases}$

⑥ $f(x) = \begin{cases} K\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-X} & x=0,1,2,3 \\ 0 & \text{o.w.} \end{cases}$

⑦ $f(x) = \begin{cases} \dfrac{K^{x}e^{-2}}{x!} & x=0,1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$

⑧ $f(x) = \begin{cases} K\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{1-x} & x=0,1 \\ 0 & \text{o.w.} \end{cases}$

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