Lecture 10: Systems of Linear Equations

Inverse Matrix Method A:

In this method, we use the concept of multiplying both sides of the system by the inverse of the matrix A, as follows:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

Finding the Inverse of Matrix A**:**

The inverse of matrix A can be found either by the traditional method (finding the adjugate matrix and dividing by the determinant) or using the row reduction method. In the row reduction method, we form an augmented matrix consisting of matrix A and the identity matrix I (of the same dimensions), and then use row operations to transform A into the identity matrix. Simultaneously, the identity matrix will transform into the inverse of A. That is:

$$[A : I] \xrightarrow{Transformation} [I : A^{-1}]$$

Example: Solve the System Using the Inverse Matrix Method

$$2x + y - z = -1$$

$$3x + 5y + 2z = 2$$

$$4x - y - 2z = 1$$

Solution:

We start by writing the augmented matrix, and then we proceed to transform matrix A into the identity matrix. The augmented matrix is:

$$[A:I]\begin{bmatrix} 2 & 1 & -1 & : & 1 & 0 & 0 \\ 3 & 5 & 2 & : & 0 & 1 & 0 \\ 4 & -1 & -2 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1'=\frac{1}{2}r_1} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 3 & 5 & 2 & : & 0 & 1 & 0 \\ 4 & -1 & -2 & : & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 3 & 5 & 2 & : & 0 & 1 & 0 \\ 4 & -1 & -2 & : & 0 & 0 & 1 \end{bmatrix} \xrightarrow[r_3'=-4r_1+r_3]{r_2'=-3r_1+r_2} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{2} & \frac{7}{2} & : & -\frac{3}{2} & 1 & 0 \\ 0 & -3 & 0 & : & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{2} & \frac{7}{2} & : & -\frac{3}{2} & 1 & 0 \\ 0 & -3 & 0 & : & -2 & 0 & 1 \end{bmatrix} \xrightarrow[r_3''=r_2']{r_2''=r_3'} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & -3 & 0 & : & -2 & 0 & 1 \\ 0 & \frac{7}{2} & \frac{7}{2} & : & -\frac{3}{2} & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & -3 & 0 & : & -2 & 0 & 1 \\ 0 & \frac{7}{2} & \frac{7}{2} & : & -\frac{3}{2} & 1 & 0 \end{bmatrix} \xrightarrow{r_2'''=-\frac{1}{3}r_2''} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & \frac{7}{2} & \frac{7}{2} & : & -\frac{3}{2} & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & : & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & \frac{7}{2} & \frac{7}{2} & : & -\frac{3}{2} & 1 & 0 \end{bmatrix} \xrightarrow[r_3'''=-\frac{7}{2}r_2'''+r_3'']{r_2''=-\frac{1}{2}r_2'''+r_1'} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & : & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{7}{2} & : & -\frac{23}{6} & 1 & \frac{7}{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & : & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{7}{2} & : & -\frac{23}{6} & 1 & \frac{7}{6} \end{bmatrix} \xrightarrow{r_3''''=\frac{2}{7}r_3'''} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & : & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & : & -\frac{23}{21} & \frac{2}{7} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & : & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & \frac{7}{2} & : & -\frac{23}{6} & 1 & \frac{7}{6} \end{bmatrix} \xrightarrow{r_1'''=\frac{1}{2}r''''_3+r_1''} \begin{bmatrix} 1 & 0 & 0 & : & -\frac{8}{21} & \frac{1}{7} & \frac{1}{6} \\ 0 & 1 & 0 & : & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & : & -\frac{23}{21} & \frac{2}{7} & \frac{1}{3} \end{bmatrix}$$

The inverse of the matrix A is:

$$A^{-1}=\begin{bmatrix} -\frac{8}{21} & \frac{1}{7} & \frac{1}{3} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ -\frac{23}{21} & \frac{2}{7} & \frac{1}{3} \end{bmatrix}=\frac{1}{21}\begin{bmatrix} -8 & 3 & 7 \\ 14 & 0 & -7 \\ -23 & 6 & 7 \end{bmatrix}$$

Thus, the solution to this system is as follows:

$$\Rightarrow\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{21}\begin{bmatrix} -8 & 3 & 7 \\ 14 & 0 & -7 \\ -23 & 6 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Therefore, the solution to the system is:

$$x=1, y=-1, \mathrm{z}=2$$

Factorization Triangular Method:

The idea of this method is to decompose a matrix A into two triangular matrices: a lower triangular matrix L and an upper triangular matrix U, with the condition that the diagonal elements of one of the matrices are equal to 1.

Given the matrix A , we can write it as:

$$A = LU$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & \mathrm{u}_{12} & \mathrm{u}_{13} \\ 0 & 1 & \mathrm{u}_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}\mathrm{u}_{12} & l_{11}\mathrm{u}_{13} \\ l_{21} & l_{21}\mathrm{u}_{12} + l_{22} & l_{21}\mathrm{u}_{13} + l_{22}\mathrm{u}_{23} \\ l_{31} & l_{31}\mathrm{u}_{12} + l_{32} & l_{31}\mathrm{u}_{13} + l_{32}\mathrm{u}_{23} + l_{33} \end{bmatrix}$$

In other words, the elements of the resulting matrix are the result of multiplying the two matrices L and U . Thus, matrix A is equal to the elements corresponding to it in the matrices L and U . One can find the elements of the two matrices by comparing their corresponding elements, where values for l_{ij} and u_{ij} can be determined sequentially. As we will see below in the equations:

$$l_{11} = a_{11}$$

$$l_{21} = a_{21}$$

$$l_{31} = a_{31}$$

$$l_{11}\mathrm{u}_{12} = a_{12} \Rightarrow \mathrm{u}_{12} = \frac{a_{12}}{l_{11}}$$

$$l_{11}u_{13} = a_{13} \Rightarrow u_{13} = \frac{a_{13}}{l_{11}}$$

$$l_{21}u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21}u_{12}$$

$$l_{31}u_{12} + l_{32} = a_{32} \Rightarrow l_{32} = a_{32} - l_{31}u_{12}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \Rightarrow u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

L,U

x, y, A

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33} \Rightarrow l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

"After finding the elements of the two matrices L,U , we rewrite the system of linear equations as follows:-

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \\ & & \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then, the imaginary vector y is found using the forward substitution method as follows:

$$Ly = b \Rightarrow \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \\ & & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} ....(1)$$

Finally, the vector x of unknowns is found using the backward substitution method as follows:

$$Uy = y \Rightarrow \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} ....(2)$$

"Where y is a fictitious vector and an unknown value. This means that the solution requires two stages: the first is to find the value of the vector y using the forward substitution method because the matrix L is a lower triangular matrix. Then, the value of the vector x is found using the backward substitution method.

To clarify the concept of the triangular factorization of the matrix A, refer to the following example:

"Example:

Find the solution to the system below using the triangular Factorization method

$$x_1 + 5x_2 + 3x_3 = 22$$
$$3x_1 + 19x_2 + 17x_3 = 94$$
$$8x_1 + 36x_2 + 25x_3 = 166$$

solution: Note that:

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 19 & 17 \\ 8 & 36 & 25 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Therefore, by performing the substitution between the elements of the two matrices above, we obtain

$$l_{11} = a_{11} = 1$$

$$l_{21} = a_{21} = 3$$

$$l_{31} = a_{31} = 8$$

$$l_{11}\mathrm{u}_{12} = a_{12} = 5$$

$$\Rightarrow \mathrm{u}_{12} = \frac{a_{12}}{l_{11}} = \frac{5}{1} = 5$$

$$l_{11}\mathrm{u}_{13} = a_{13} = 3$$

$$\Rightarrow \mathrm{u}_{13} = \frac{a_{13}}{l_{11}} = \frac{3}{1} = 3$$

$$l_{21}\mathrm{u}_{12} + l_{22} = a_{22} = 19$$

$$\Rightarrow l_{22} = a_{22} - l_{21}\mathrm{u}_{12} = 19 - 3\times 5 = 4$$

$$l_{31}\mathrm{u}_{12} + l_{32} = a_{32} = 36$$

$$\Rightarrow l_{32} = a_{32} - l_{31}\mathrm{u}_{12} = 36 - 8\times 5 = -4$$

$$l_{21}\mathrm{u}_{13} + l_{22}\mathrm{u}_{23} = a_{23} = 17$$

$$\Rightarrow \mathrm{u}_{23} = \frac{a_{23} - l_{21}\mathrm{u}_{13}}{l_{22}} = \frac{17 - 3\times 3}{4} = 2$$

$$l_{31}\mathrm{u}_{13} + l_{32}\mathrm{u}_{23} + l_{33} = a_{33} = 25$$

$$\Rightarrow l_{33} = a_{33} - l_{31}\mathrm{u}_{13} - l_{32}\mathrm{u}_{23} = 25 - 8\times 3 - (-4)\times 2 = 9$$

Therefore, we have:

$$LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \\ & & \end{bmatrix} \begin{bmatrix} 1 & \mathrm{u}_{12} & \mathrm{u}_{13} \\ 0 & 1 & \mathrm{u}_{23} \\ 0 & 0 & 1 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 8 & -4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we use the above equation (Equation 1) to find the value of the vector y:

$$Ly = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 8 & -4 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 94 \\ 166 \end{bmatrix}$$

Using forward substitution, we get:

$$y_1 = 22$$

$$3y_1 + 4y_2 = 94$$

$$\Rightarrow y_2 = \frac{94 - 3y_1}{4} = \frac{94 - 3 \times 22}{4} = 7$$

$$8y_1 - 4y_2 + 9y_3 = 166$$

$$\Rightarrow y_3 = \frac{166 - 8y_1 - 4y_2}{9} = \frac{166 - 8 \times 22 + 4 \times 7}{9} = 2$$

Now, we use equation (2) to find the value of the vector x as follows:

$$Ux = y = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 7 \\ 2 \end{bmatrix}$$

Using backward substitution, we get:

$$x_3 = 2$$
$$x_2 + 2x_3 = 7$$
$$\Rightarrow x_2 = 7 - 2x_3 = 7 - 2 \times 2 = 3$$
$$x_1 + 5x_2 + 3x_3 = 22$$
$$\Rightarrow x_1 = 22 - 5x_2 - 3x_3 = 22 - 5 \times 3 - 3 \times 2 = 1$$

Therefore, the solution to the system of linear equations in the example is as follows:

$$x_1 = 1 \text{ , } x_2 = 3 \text{ , } x_3 = 2$$