Lecture 3: Linear and Nonlinear Equations:

Linear Equation (also called the equation of a straight line): A linear equation is any equation that can be graphically represented as a straight line, and it is written in the form:

$$f(x) = ax + b \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Nonlinear Equation: A nonlinear equation is any equation that cannot be graphically represented as a straight line; instead, it is represented as a curve. A nonlinear equation contains at least one nonlinear term (examples include, $x^2, \log x, \sin x, e^x$). Some nonlinear equations can be easily solved, while many others are very complex or even impossible to solve analytically. Therefore, numerical methods must often be employed to solve them.

Solving Equations: By solving an equation, we mean finding the root of the function $f(x)$, where the root is the value of x that makes the function $f(x)$ equal to zero. That is, the root of the function is the value that satisfies the following equation:

$$f(x) = 0$$

Solving Linear Equations: Notice that the root of the linear function shown in equation (1) above is always in the form $f(x) = \frac{-b}{a}$. This can be derived by setting equation (1) equal to zero and solving for x as follows:

$$f(x) = ax + b = 0$$
$$\rightarrow ax = -b$$
$$\rightarrow f(x) = -\frac{b}{a}$$

Solving Nonlinear Equations: solving nonlinear equations means finding the root of the equation that satisfies the value of the equation equals zero. Some nonlinear equations have only one root, for example: $f(x) = x^3 - 8$ This equation has one root, $x = 2$, which can be found by setting the equation equal to zero, moving 8 to the right-hand side, and then taking the cube root of both sides, as follows:

$$f(x) = x^3 - 8 = 0$$
$$x^3 = 8$$
$$x = 2$$

While some nonlinear equations have more than one root, such as $x^2 = 4$, it has two roots, $x = 2$ and $x = -2$.

note that in the examples above, finding the roots was relatively easy. Now, suppose we want to find a root (or roots) of the following equation:

$$f(x) = 3x^3 - 4x + 1 \qquad \text{.......................... (2)}$$

In this case, it is clear that solving the above equation is not as easy as in the previous examples. We will see later that the function $f(x)$ defined in equation (2) above has three roots, which are: $x = -1.263763$, or $x = 0.263763$, or $x = 1$

Graphical Method for Determining the Roots of Nonlinear Equations:

The graphical method can be used to approximate the roots of a nonlinear equation. The root of the equation is represented by the point where the function curve intersects the x-axis, i.e., where the y-axis (which represents the function value) is equal to zero.

You can plot the nonlinear function manually by substituting various values for the variable x into the function and then finding the function's value at these points to obtain pairs of data in the form $(x, f(x))$ or (x, y). Afterward $y = f(x)$, you plot these pairs on a graph paper and draw a smooth curve through the plotted points. This method is slow and inefficient for determining the root value manually, so it is preferable to use computational software (such as R or MATLAB) to accurately plot the function and identify the roots.

Example: Use the given values of x to find the function value as illustrated in equation (2) above. Then use the calculated points to plot the function graphically and approximately determine the values of x that make the function equal to zero using the graph:

$$f(x) = 3x^3 - 4x + 1$$

x	-2	-1	0	1	2
$f(x)$					

Solution: Substitute all given values of x into the function to find the corresponding y values:

$$f(x=-2)=3(-2)^3-4(-2)+1=-15$$

$$f(x=-1)=3(-1)^3-4(-1)+1=2$$

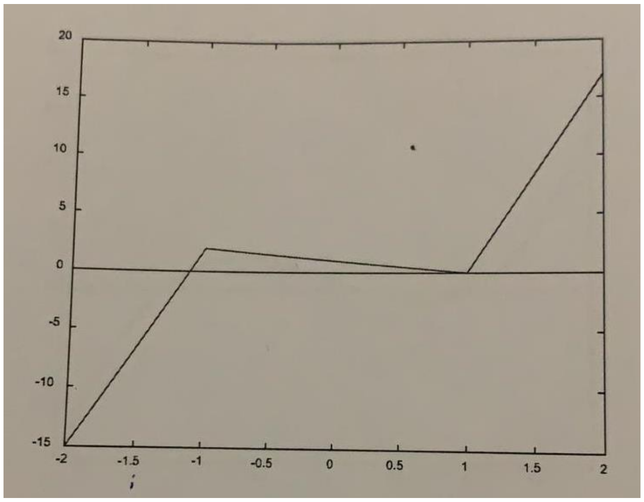
$$f(x=0)=3(0)^3-4(0)+1=1$$

$$f(x=1)=3(1)^3-4(1)+1=0$$

$$f(x=2)=3(2)^3-4(2)+1=17$$

x	-2	-1	0	1	2
$f(x)$	-15	2	1	0	17

You can then use the pairs shown in the table to plot the function as follows:



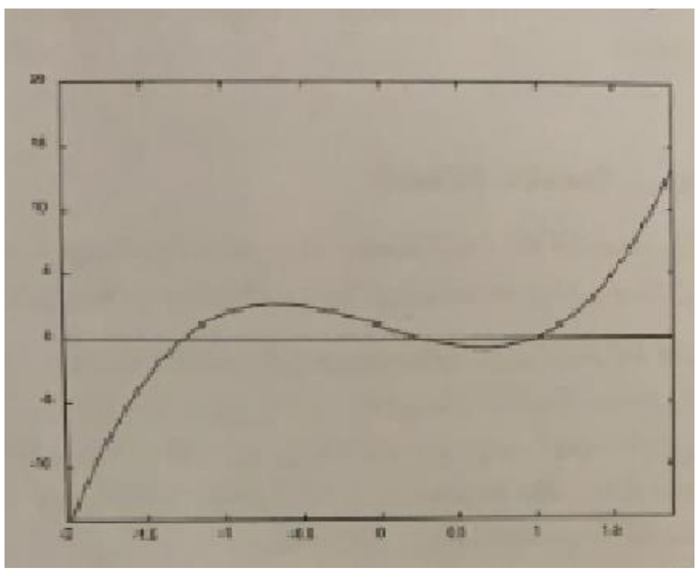
Notice that the points where the function curve intersects the x-axis (i.e., where $y=0$) represent the roots of the function $f(x)$. From the graph above, it can be inferred that the function has only two roots, approximately at $(-1.2, 1)$.

However, it is clear that the graph above is not very accurate because only five points were used. To improve accuracy, more points should be considered. This will increase the effort required to find the solution manually, so it is better to use MATLAB for accurate plotting and root determination.

To plot the function accurately, select a large number of x values within a specified range, for example, take all values between two numbers $\left[-2, 2\right]$ with an increment of 0.1. Thus, the values of x will be:

$$x = -2, 1.9, -1.8, ..., -0.2, -0.1, 0, 0.1, 0.2..., 1.8, 1.9, 2$$

After that, compute the function values for the specified x values and plot all points to form a smooth curve. This will yield a more accurate graph, as shown below.



Note that there are three roots of the function, approximately, $\left(-1.25, 0.25, 1\right)$ and the next page illustrates the commands used in MATLAB to execute the plot above.