Lecture 9: Systems of Linear Equations

Systems of Linear Equations:
We previously discussed linear equations in Lecture 3, noting that a linear equation is any equation that can be graphically represented by a straight line. This is true when there is only one variable (or unknown) in the equation, such as:

$$f\left(x\right)=ax+b$$

We also mentioned that the general solution to the linear equation can be expressed as:

$$x=-\frac{b}{a}$$

(Refer to Lecture 3).
Linear equations can contain more than one unknown (for example: $x_{1,}x_{2,}x_{3,}.....$). In such cases, it is necessary to have a system of simultaneous linear equations that must be solved together to yield the correct values for the unknowns. In this lecture and the subsequent ones, we will discuss the most common direct methods and numerical algorithms used to solve systems of linear equations simultaneously.

Let us assume we have the following system of linear equations:

$$a_{11}x_1+a_{12}x_2+.......+a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+.......+a_{2n}x_n=b_2$$

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$$a_{m1}x_1+a_{m2}x_2+.......+a_{mn}x_n=b_m$$

This means there are mmm equations and nnn unknowns. The system can be represented using matrices as follows:

$$
\begin{bmatrix} a_{11} & a_{12} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots & a_{2n} \\ & & . & \\ & & . & \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}_{m\times 1}
$$

Or, more concisely:

$$Ax = b$$

where A is the matrix of coefficients with dimensions $n \times m$, which is known beforehand, and x is the vector of unknowns of length. n. Our objective is to determine the values of this vector. The vector b is the constants on the right-hand side with length m.

Notice that the element in the a_{ij} equation, where $i = 1,2,3...m,$ and $j = 1,2,3,...n$ represents the coefficient a_{ij} of the unknown x_j, which is known beforehand. Our objective is to find the values of the unknowns $x_{1,} x_{2,} x_{3,} .....x_n$ that satisfy the above system (i.e., the values that make both sides of the equations equal).

To find a solution, we must identify the three possible cases that we may encounter before starting the solution process. These cases are:

1. **No solution**: In this case, there is no solution that satisfies the system due to its inconsistency. For example:

$$x_1 - x_2 = 8$$

$$3x_1 - 3x_2 = 10$$

Here, it is impossible to find values for x_1 and x_2 that satisfy the above system because the two equations are inconsistent, meaning they do not correspond to the same solution.

Note: A system is considered consistent if there is at least one solution that satisfies it.

2. **Infinitely many solutions**: This situation occurs when one of the equations is identical to another equation in the system. For example:

$$2x_1 + 0.7x_2 = 3$$
$$6x_1 + 2.1x_2 = 9$$

Notice that the second equation is equivalent to the first multiplied by three. Thus, the two equations are identical, indicating there are infinitely many solutions that satisfy the system. This case is referred to as a **dependent system**.

3. **Unique solution**: This situation means there is exactly one solution that satisfies the system. This occurs when the system is both **consistent and independent**, meaning there is no overlap between any two equations in the system.

The table below summarizes the three possible cases in systems of linear equations:

Number of Solutions	**Description**
No solution	Inconsistent
Infinitely many solutions	Consistent and dependent
Exactly one solution	Consistent and independent

Note: Our focus in this study will be on systems that have exactly one solution, specifically when $n = m$.

Methods for Solving Linear Systems

There are several methods for solving systems of linear equations. Some are direct methods that yield exact solutions, while others are indirect methods that provide approximate (numerical) solutions. We will cover some of these methods in this lecture and the upcoming ones. Before that, let's review some important information about matrices:

- A **square matrix** is a matrix where the number of rows equals the number of columns.
- A **diagonal matrix** D is a matrix where all its elements are zero except for the elements on the diagonal.
- A **unit matrix** I is a diagonal matrix whose main diagonal elements are equal to 1.
- An **upper triangular matrix** U is a matrix where all the elements below the main diagonal are zeros.
- A **lower triangular matrix** L is a matrix where all the elements above the main diagonal are zeros.
- The **transpose** of a matrix A is a matrix that can be formed by switching the rows and columns of A. It is denoted as A^T or A'.
- A **symmetric matrix** is a matrix that equals its transpose, meaning $\mathrm{A}^T = A$
- The **inverse** of a matrix A is a matrix that satisfies the equation: $AA^{-1} = A^{-1}A = 1$
- If the determinant of matrix A equals zero, then matrix A does not have an inverse.
- If the inverse of matrix A does not exist, then the matrix A is called a **singular matrix**.
- To add or subtract two matrices, the matrices must have the same dimensions.
- To multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second matrix.

Transformation Matrix

To transform a given matrix into a different form from its original format, we can perform some algebraic and row operations on the original matrix. These transformations include:

- Multiplying (or dividing) one row of the matrix by a non-zero constant.
- Swapping two rows.
- Multiplying (or dividing) one row by a non-zero constant and then adding the resulting row to another row.

Example: Transform the following matrix into an upper triangular matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & -5 \\ -6 & -8 & 1 \end{bmatrix}$$

Solution: To convert the matrix A into an upper triangular form, we need to perform several transformations and linear operations on the rows to achieve zeros in the lower triangle (below the main diagonal). The steps are as follows:

1. Multiply the first row by (-2) and add the result to the second row:

$$r_2' = -2r_1 + r_2$$
$$= -2\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -3 & -5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -7 & -11 \end{bmatrix}$$

Now the matrix looks like this:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -11 \\ -6 & -8 & 1 \end{bmatrix}$$

2. Next, we will modify the third row by multiplying the first row by 6 and adding it to the third row:

$$\begin{aligned} r_3' &= 6r_1 + r_3 \\ &= 6\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} -6 & -8 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 19 \end{bmatrix} \end{aligned}$$

Now the matrix becomes:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -11 \\ 0 & 4 & 19 \end{bmatrix}$$

3. Finally, we can continue transforming the matrix by eliminating the entry in the third row's second column. Multiply the second row by $\frac{4}{7}$ and add it to the third row:

$$r_3'' = \frac{4}{7} r_2' + r_3'$$

$$= \frac{4}{7}\begin{bmatrix}0 & -7 & -11\end{bmatrix} + \begin{bmatrix}0 & 4 & 19\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & \frac{89}{7}\end{bmatrix}$$

At this point, the transformed upper triangular matrix is:

$$A = \begin{bmatrix}1 & 2 & 3 \\ 0 & -7 & -11 \\ 0 & 0 & \frac{89}{7}\end{bmatrix}$$

The process continues until the matrix is fully transformed into upper triangular form.

Solving Systems of Linear Equations – Direct Methods

Gaussian Elimination Method

The idea behind this method is to form the augmented matrix $[A : \mathrm{b}]$ and then convert the matrix A into an upper triangular matrix using row operations. After that, the system is solved using **back substitution**. To understand the method more clearly, let's consider the following example:

Example: Find the solution to the following system using Gaussian elimination:

$$2x + y + z = 7$$
$$4x + 4y + 3z = 21$$
$$6x + 7y + 4z = 32$$

Solution: We start by forming the augmented matrix as follows:

$$[A : \mathrm{b}] = \begin{bmatrix} 2 & 1 & 1 : & 7 \\ 4 & 4 & 3 : & 21 \\ 6 & 7 & 4 : & 32 \end{bmatrix}$$

Now we begin performing row operations to convert the matrix A into an upper triangular form:

1. Multiply the first row by (-2) and add it to the second row:

$$\begin{aligned} r_2' &= -2r_1 + r_2 \\ &= -2[2 \quad 1 \quad 1 : 7] + [4 \quad 4 \quad 3 : 21] \\ &= [0 \quad 2 \quad 1 : \ 7] \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 : & 7 \\ 4 & 4 & 3 : & 21 \\ 6 & 7 & 4 : & 32 \end{bmatrix} \xrightarrow{r_2' = -2r_1 + r_2} \begin{bmatrix} 2 & 1 & 1 : & 7 \\ 0 & 2 & 1 : & 7 \\ 6 & 7 & 4 : & 32 \end{bmatrix}$$

2. We multiply the first row by (-3) and add it to the third row:

$$r_3' = -3r_1 + r_3$$
$$= -3\begin{bmatrix}2 & 1 & 1:7\end{bmatrix} + \begin{bmatrix}6 & 7 & 4:32\end{bmatrix}$$
$$= \begin{bmatrix}0 & 4 & 1: & 11\end{bmatrix}$$

Now the matrix becomes:

$$\begin{bmatrix}2 & 1 & 1: & 7\\ 0 & 2 & 1: & 7\\ 6 & 7 & 4: & 32\end{bmatrix} \xrightarrow{r_3'=-3r_1+r_3} \begin{bmatrix}2 & 1 & 1: & 7\\ 0 & 2 & 1: & 7\\ 0 & 4 & 1: & 11\end{bmatrix}$$

3. multiplying the second row by (-2) and adding it to the third row:

$$r_3'' = -2r_2' + r_3'$$
$$= -2\begin{bmatrix}0 & 2 & 1:7\end{bmatrix} + \begin{bmatrix}0 & 4 & 1:11\end{bmatrix}$$
$$= \begin{bmatrix}0 & 0 & \text{-}1: & \text{-}3\end{bmatrix}$$

$$\begin{bmatrix}2 & 1 & 1: & 7\\ 0 & 2 & 1: & 7\\ 0 & 4 & 1: & 11\end{bmatrix} \xrightarrow{r_3''=-2r_2'+r_3'} \begin{bmatrix}2 & 1 & 1: & 7\\ 0 & 2 & 1: & 7\\ 0 & 0 & \text{-}1: & \text{-}3\end{bmatrix}$$

Now we have an upper triangular matrix.

$$\begin{bmatrix}A:\text{b}\end{bmatrix} \xrightarrow{transformations} \begin{bmatrix}2 & 1 & 1: & 7\\ 0 & 2 & 1: & 7\\ 0 & 0 & \text{-}1: & \text{-}3\end{bmatrix}$$

After transforming the system, we have the following equations:

$$x + y + z \ = 7$$
$$2y + z = 7$$
$$-z = -3$$

Now, we can apply back substitution, starting from the last equation:

$$-z = -3$$
$$\Rightarrow z = 3$$

Next, we substitute $z = 3$ into the second equation:

$$\Rightarrow 2y + z = 7$$
$$\Rightarrow y = 2$$

we substitute both $y = 2, z = 3$ into the first equation:

$$2x + y + z \ = 7$$
$$\Rightarrow 2x + 2 + 3 = 7$$
$$\Rightarrow x = 1$$

Thus, the solution to the system is:

$$\begin{bmatrix} x & \text{y} & \text{z} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

This completes the process of solving the system of equations using Gaussian elimination and back substitution.

Gauss-Jordan Elimination Method

This method is quite similar to the Gaussian elimination method, where the augmented matrix is also formed. However, the difference here is that matrix A is transformed into the identity matrix instead of an upper triangular matrix. When matrix A is converted into the identity matrix, the vector b directly turns into the solution. This can be represented as:

$$[A:\mathrm{b}] \xrightarrow{transformations} [I:\mathrm{x}]$$

Where:

$$[A:\mathrm{b}] = \begin{bmatrix} a_{11} & a_{12} & \dots\dots\dots & a_{1n} & : \mathrm{b}_1 \\ a_{21} & a_{22} & \dots\dots\dots & a_{2n} & : \mathrm{b}_2 \\ & & . & & : . \\ & & . & & : . \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{nn} & : \mathrm{b}_n \end{bmatrix} \xrightarrow{transformations} \begin{bmatrix} 1 & 0 & \dots\dots & 0 : & x_1 \\ 0 & 1 & \dots\dots & 0 : & x_2 \\ . & . & . & : & . \\ . & . & . & : & . \\ 0 & 0 & \dots\dots & 1 : & x_n \end{bmatrix} = [\mathrm{I}:\mathrm{x}]$$

Example

Find the solution to the following system using the Gauss-Jordan elimination method:

$$x + y + z = 3$$

$$2x + 3y + 7z = 0$$

$$x + 3y - 2z = 17$$

Solution: We begin by writing the augmented matrix:

$$[A:\mathrm{b}] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 2 & 3 & 7 & : & 0 \\ 1 & 3 & -2 & : & 17 \end{bmatrix} \xrightarrow[r_3' = -r_1 + r_3]{r_2' = -2r_1 + r_2} \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 5 & : & -6 \\ 0 & 2 & -3 & : & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 1 & 5 & : & -6 \\ 0 & 2 & -3 & : & 14 \end{bmatrix} \xrightarrow[r_3''=-2r_2'+r_3']{r_1'=-r_2'+r_1} \begin{bmatrix} 1 & 0 & -4 & : & 9 \\ 0 & 1 & 5 & : & -6 \\ 0 & 0 & -13 & : & 26 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & : & 9 \\ 0 & 1 & 5 & : & -6 \\ 0 & 0 & -13 & : & 26 \end{bmatrix} \xrightarrow{r_3'''=\frac{1}{13}r_3''} \begin{bmatrix} 1 & 0 & -4 & : & 9 \\ 0 & 1 & 5 & : & -6 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & : & 9 \\ 0 & 1 & 5 & : & -6 \\ 0 & 0 & 1 & : & -2 \end{bmatrix} \xrightarrow[r_2''=-5r_3'''+r_2'']{r_1''=4r_3'''+r_1'} \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 4 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$$

Thus, the solution to this system is:

$$\begin{bmatrix} x & \mathrm{y} & \mathrm{z} \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \end{bmatrix}$$

you can verify this by substituting these values back into the original system to ensure that they yield the correct results on the right-hand side: