Lecture 7: Methods for Solving Nonlinear Equations Numerically - Newton-Raphson Algorithm

The **Newton-Raphson** algorithm was developed by both Isaac Newton and Joseph Raphson in the 17th century. This algorithm is used to find the roots of real functions through a series of iterative steps. The concept of the Newton-Raphson algorithm is based on using the equation of the tangent line to the curve of the function to approximate the root value. However, the function $f(x)$ must be a real and differentiable function. The mechanism of the algorithm can be summarized as follows:

First, we define the allowable error value (ε epsilon) and then select an initial guess, say x_1. We plot the function and identify the point $p_1 = (x_1, f(x_1))$ on the graph. Next, we draw a straight line that touches the function curve at point p_1. This line is referred to as the **tangent line** to the curve of the function or simply the **tangent** (see the illustration in the example below).

Let $(x_2, 0)$ the point where this tangent line intersects the x-axis be denoted as x_2 becomes a candidate for the root. We then check whether $|x_2 - x_1| \leq \varepsilon$ or $|f(x_2)| \leq \varepsilon$ If the condition is satisfied, then x_2 represents the required root. Otherwise, the point $p_2 = (x_2, f(x_2))$ is plotted, and a new tangent line is drawn at point $(x_3, 0)$, which intersects the x-axis at a new point x_3

We continue this process, testing at each step whether $|x_n - x_{n-1}| \leq \varepsilon|$ or $|f(x_n)| \leq \varepsilon$ the required root x_n is found.

Deriving the Formula for Root Calculation:

Let x_1 be the initial guess for the root of the equation $f(x)=0$, where f is a real differentiable function. We plot the function and identify the point $p_1=(x_1, f(x_1))$.then we plot straight line tangent to the curve of the function at the point p_1, (see the illustration in the example below).

Now, let $(x_2, 0)$ be the point where the tangent line intersects the x-axis. The slope of the tangent line between the points $(x_1, f(x_1))$ and $(x_2, 0)$ can be calculated using the formula below:

$$slope = \frac{\Delta y}{\Delta x} = \frac{0 - f(x_1)}{x_2 - x_1}$$

We know that the slope is approximately equal to the derivative of the function, so:

$$f'(x_1) = \frac{0 - f(x_1)}{x_2 - x_1} = -\frac{f(x_1)}{x_2 - x_1} \quad \text{.........(1)}$$

Where $f'(x_1) = \dfrac{df(x)}{dx}$ is the first derivative of the function with respect to x.

Now, solving equation (1) for x_2, we get:

$$f'(x_1) = -\frac{f(x_1)}{x_2 - x_1}$$

$$\Rightarrow x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, the $n-th$ root can be calculated using the formula below:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

provided that $f'(x_1) \neq 0$, where n=2,3,4,…

Steps of the Newton-Raphson Algorithm:

1. Determine the allowable error value ε epsilon.
2. Choose an initial guess for the root, let it be x_1.
3. Find the derivative of the function $f'(x_1) = \frac{df(x)}{dx}$.
4. Set $n = 2$.
5. Calculate the root using the formula: $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$
6. Check if $|x_n - x_{n-1}| \leq \varepsilon$ or. $|f(x_n)| \leq \varepsilon$ If either condition is met, x_n is the root.

7. Otherwise, if $\left|x_n - x_{n-1}\right| > \varepsilon$ and $\left|f\left(x_n\right)\right| > \varepsilon$, then set $n = n+1$ and go back to step 5.

Example: Find the root of the function $f\left(x\right) = x - e^{-x} = 0$ using the Newton-Raphson algorithm. Use $\varepsilon = 0.0001$.

Solution:

- **Step 1:** The allowed error $\varepsilon = 0.0001$.
- **Step 2:** Assume the initial guess for the root is $x_1 = 0$.
- **Step 3:** Find the derivative of the function:

$$f'\left(x\right) = \frac{df}{dx}\left(x - e^{-x}\right)$$
$$= 1 + e^{-x}$$

- **Step 4:** Set $n = 2$.
- **Step 5:** Calculate the next approximation for the root

$$x_n = x_{n-1} - \frac{f\left(x_{n-1}\right)}{f'\left(x_{n-1}\right)}$$

$$
\begin{aligned}
\Rightarrow x_2 - x_1 &= -\frac{f(x_1)}{f'(x_1)} \\
&= 0 - \frac{f(0)}{f'(0)} \\
&= 0 - \frac{0 - e^{-0}}{1 + e^{-0}} \\
&= \frac{1}{1+1} = 0.5
\end{aligned}
$$

Step 6: Test if the proposed value represents a root of the function or not:

$$
|f(x_2)| = |f(0.5)| = \left|0.5 - e^{-0.5}\right| = |-0.1065| > \varepsilon
$$

Since, x_2 is not the required root. Therefore, we need to perform another iteration to find x_3.

The next iteration is calculated as:

$$x_n = x_{n-1} - \frac{f\left(x_{n-1}\right)}{f'\left(x_{n-1}\right)}$$

$$\Rightarrow x_3 = x_2 - \frac{f\left(x_2\right)}{f'\left(x_2\right)}$$

$$= 0.5 - \frac{f\left(0.5\right)}{f'\left(0.5\right)}$$

$$= 0.5 - \frac{0.5 - e^{-0.5}}{1 + e^{-0.5}}$$

$$= 0.5 - \frac{-0.1065}{1.6065} = 0.5 + 0.0663 = 0.5663$$

Note that:

$$\left| f\left(x_3\right) \right| = \left| f\left(0.5663\right) \right| = \left| 0.5663 - e^{-0.5663} \right| = \left| -0.10013 \right| > \varepsilon$$

Since x_3 is not the required root. Therefore, another iteration is needed to find x_4.

The next iteration is calculated as:

$$x_n = x_{n-1} - \frac{f\left(x_{n-1}\right)}{f'\left(x_{n-1}\right)}$$

$$\Rightarrow x_4 = x_3 - \frac{f\left(x_3\right)}{f'\left(x_3\right)}$$

$$= 0.5663 - \frac{f\left(0.5663\right)}{f'\left(0.5663\right)}$$

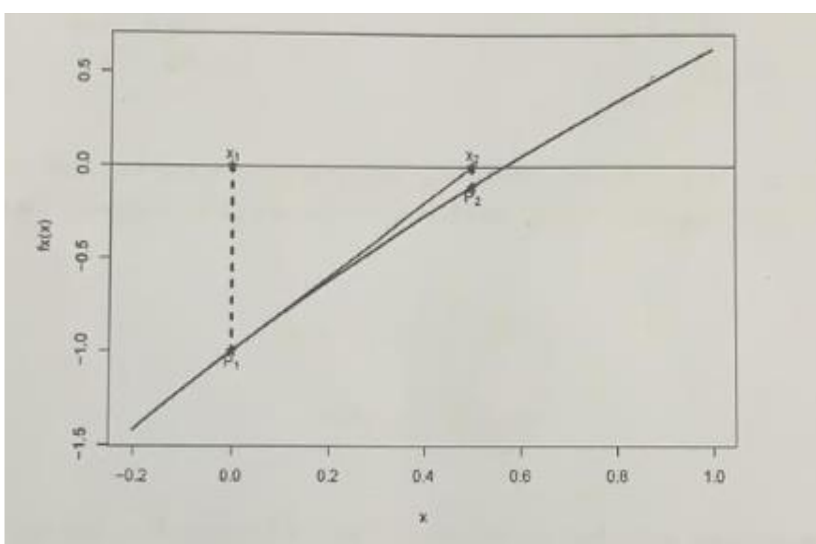
$$= 0.5663 - \frac{0.5663 - e^{-0.5663}}{1 + e^{-0.5663}}$$

$$= 0.5663 - \frac{-0.0013}{1.5676} = 0.5663 + 0.0008 = 0.5671$$

Now, let's check if this value satisfies the condition for the root:

$$\left| f\left(x_4\right) \right| = \left| f\left(0.5671\right) \right| = \left| 0.5671 - e^{-0.5671} \right| = \left| -0.0001 \right| = \varepsilon$$

Since $x_4 = 0.5671$ is the required root or solution.



Using the Newton-Raphson Algorithm to Find Different Roots of Real Numbers:

Here, we refer to roots such as square roots, cube roots, or roots of any other order. The Newton-Raphson algorithm can be used to find the r^{th} root of any positive real number a. In other words, we want to find the value of x such that:

$$x = \sqrt[r]{a}$$

Where a is a positive real number for which we wish to find the root, and i r s the order of the root.

Notice that:

$$x = \sqrt[r]{a}$$

$$\Rightarrow x = a^{\frac{1}{r}}$$

$$\Rightarrow x^r = a$$

$$\Rightarrow x^r - a = 0$$

Examples:

$x = \sqrt[2]{16}$ $\Rightarrow x = 16^{\frac{1}{2}}$ $\Rightarrow x^2 = 16$ $\Rightarrow x^2 - 16 = 0$	$x = \sqrt[3]{46.72}$ $\Rightarrow x = 46.72^{\frac{1}{3}}$ $\Rightarrow x^3 = 46.72$ $\Rightarrow x^3 - 46.72 = 0$

$x = \sqrt[7]{88}$ $\Rightarrow x = 88^{\frac{1}{7}}$ $\Rightarrow x^7 = 88$ $\Rightarrow x^3 - 88 = 0$	$x = \sqrt[1.5]{21}$ $\Rightarrow x = 21^{\frac{1}{1.5}}$ $\Rightarrow x^{1.5} = 21$ $\Rightarrow x^3 - 21 = 0$

Let $f(x) = x^r - a = 0$ To apply the **Newton-Raphson method** to find approximate value of x that makes the value of the function equal to zero, note that $f'(x) = rx^{r-1}$ derivative of the function we start with the general equation:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$= x_{n-1} - \frac{x_{n-1}^r - a}{rx_{n-1}^{r-1}}$$

You can apply the Newton-Raphson algorithm using the final formula iteratively to estimate the value of the r^{th} root of the positive real number a:

Example 4:

Calculate the fourth root of the number 27, meaning find the value of x that satisfies $x = \sqrt[4]{27}$ Note that: the required accuracy to four decimal places

Solution:

$$x = \sqrt[4]{27}$$

$$\Rightarrow x = 27^{\frac{1}{4}}$$

$$\Rightarrow x^4 = 27$$

$$\Rightarrow x^3 - 27 = 0$$

the function will be $f(x) = x^4 - 27$ and The derivative formula as:

$$f'(x) = 4x^3$$

Now we can apply the Newton-Raphson method to find the value of x that satisfies:

$f(x) = x^4 - 27 = 0$ as:

Step 1: From the question, the required accuracy to four decimal places means that $\varepsilon = 0.0001 = 1 \times 10^{-4}$

Step 2: Assume $x_1 = 2$.

Step 3: The derivative will be:

$$f'(x) = 4x^3$$

Step 4: Let $n = 2$.

Step 5: Calculate the root value:

$$x_n = x_{n-1} - \frac{f\left(x_{n-1}\right)}{f'\left(x_{n-1}\right)}$$

$$x_2 = x_1 - \frac{f\left(x_1\right)}{f'\left(x_1\right)}$$

$$= 2 - \frac{f\left(2\right)}{f'\left(2\right)}$$

$$= 2 - \frac{2^4 - 27}{4(2^3)}$$

$$= 2 - \frac{-11}{32} = 2.3438$$

Step 6: Test the proposed value:

$$\left| f\left(x_2 = 2.3438\right)\right| = \left|2.3438^4 - 27\right| = \left|3.1774\right| > \varepsilon$$

$$\left|x_2 - x_1\right| = \left|2.3438 - 2\right| = \left|0.3438\right| > \varepsilon$$

So, we need another iteration.

$$x_3 = x_2 - \frac{f\left(x_2\right)}{f'\left(x_2\right)}$$

$$= 2.3438 - \frac{f\left(2.3438\right)}{f'\left(2.3438\right)}$$

$$= 2.3438 - \frac{2.3438^4 - 27}{4(2.3438^3)}$$

$$= 2.3438 - \frac{3.1774}{51.5017} = 2.2821$$

$$Test\ the$$

$$\left|f\left(x_3 = 2.2821\right)\right| = \left|2.2821^4 - 27\right| = \left|0.1231\right| > \varepsilon$$

$$\left|x_3 - x_2\right| = \left|2.2821 - 2.3438\right| = \left|-0.0617\right| > \varepsilon$$

New Iteration

$$x_4 = x_3 - \frac{f\left(x_3\right)}{f'\left(x_3\right)}$$

$$= 2.2821 - \frac{f\left(2.2821\right)}{f'\left(2.2821\right)}$$

$$= 2.2821 - \frac{2.2821^4 - 27}{4(2.2821^3)}$$

$$= 2.3438 - \frac{0.1231}{47.5405} = 2.2795$$

Now we test again:

$$\left| f\left(x_4 = 2.2795\right)\right| = \left|2.2795^4 - 27\right| = \left|-0.0003\right| > \varepsilon$$

$$\left|x_4 - x_3\right| = \left|2.2795 - 2.2821\right| = \left|-0.0026\right| > \varepsilon$$

New Iteration

$$x_5 = x_4 - \frac{f\left(x_4\right)}{f'\left(x_4\right)}$$

$$= 2.2795 - \frac{f\left(2.2795\right)}{f'\left(2.2795\right)}$$

$$= 2.2795 - \frac{2.2795^4 - 27}{4(2.2795^3)}$$

$$= 2.2795 - \frac{-0.0003}{47.3782} = 2.2795$$

$$\left| f\left(x_5 = 2.2795\right)\right| = \left|2.2795^4 - 27\right| = \left|-0.0003\right| > \varepsilon$$

$$\left|x_5 - x_4\right| = \left|2.2795 - 2.2795\right| = \left|0\right| < \varepsilon$$

Since one of the above conditions has been satisfied, we can stop and conclude that the approximate value of $x = 2.2795$ that makes the equals zero. $f\left(x\right) = x^4 - 27$. In other words ,

$$x = \sqrt[4]{27} = 2.2795$$