INTEGRAL METHODS\ TRIGOMETRIC SUB.

$\sqrt{a^2+u^2}$, $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$: في هذه الحالة نكامل الدوال من الصيغ الاتية

NOTES :

$$\int \tan x dx = -Ln\left|\cos x\right|$$

$$\int \cot x dx = Ln\left|\sin x\right|$$

$$\int \sec x dx = Ln\left|\sec x + \tan x\right|$$

$$\int \csc x dx = -Ln\left|\csc x + \cot x\right|$$

في حالة اعطاء دالة تحوي احدى الصيغ الثلاث اعلاه فانه يتم التعويض بالفرضيات التالية :

$$\sqrt{a^2+u^2} \Rightarrow u = a\tan\theta$$

$$\sqrt{a^2-u^2} \Rightarrow u = a\sin\theta$$

$$\sqrt{u^2-a^2} \Rightarrow u = a\sec\theta$$

EXAM :

$\boxed{1}$ $\int \frac{du}{16+u^2}$

$$u = 4\tan\theta \Rightarrow du = 4\sec^2\theta d\theta \quad , \quad \tan\theta = \frac{u}{4} \Rightarrow \theta = \tan^{-1}(\frac{u}{4})$$

$$\int \frac{du}{16+u^2} = \int \frac{4\sec^2\theta d\theta}{16+16\tan^2\theta}$$

$$= \frac{1}{4}\int \frac{\sec^2\theta d\theta}{1+\tan^2\theta}$$

$$= \frac{1}{4}\int \frac{\cancel{\sec^2\theta} d\theta}{\cancel{\sec^2\theta}} = \frac{1}{4}\int d\theta = \frac{1}{4}\theta = \frac{1}{4}\tan^{-1}(\frac{u}{4}) + C$$

$\boxed{2}$ $\int \frac{dx}{x^2\sqrt{4-x^2}}$

$$x = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta \quad , \ \sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}(\frac{x}{2})$$

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}$$

$$= \frac{1}{4}\int \frac{\cancel{2\cos}\theta d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta d\theta = \frac{-1}{4}\cot\theta = \frac{-1}{4}\cot(\sin^{-1}(\frac{x}{2})) + C$$

$\boxed{3}$ $\int \frac{\sqrt{x^2-25}}{x}dx$

$$x = 5\sec\theta \Rightarrow dx = 5\sec\theta\tan\theta d\theta \quad , \ \sec\theta = \frac{x}{5} \Rightarrow \theta = \sec^{-1}(\frac{x}{5})$$

$$= \int \frac{\sqrt{25\sec^2\theta - 25}}{\cancel{5\sec\theta}} \cancel{5\sec\theta} \tan\theta d\theta$$

$$= 5\int \sqrt{\sec^2\theta - 1} \cdot \tan\theta d\theta$$

$$= 5\int \tan^2\theta d\theta = \int \left(\sec^2\theta - 1\right) d\theta$$

$$= 5[\tan\theta - \theta] = 5\left[\tan\sec^{-1}(\frac{x}{5}) - \sec^{-1}(\frac{x}{5})\right] + C$$

$\boxed{4}$ $\int_0^2 \sqrt{4-x^2}dx$

$x = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta$, $\sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}(\frac{x}{2})$

$x = 0 \Rightarrow \theta = 0$, $x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^2 \sqrt{4-x^2}dx = \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta d\theta = \int_0^{\frac{\pi}{2}} 4\cos^2\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta) d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = ?$$

$\boxed{5}$ $\int_0^3 \frac{x^2}{\sqrt{9-x^2}}dx$

$x = 3\sin\theta \Rightarrow dx = 3\cos\theta d\theta$, $\sin\theta = \frac{x}{3} \Rightarrow \theta = \sin^{-1}\frac{x}{3}$

$x = 0 \Rightarrow \theta = 0$, $x = 3 \Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^3 \frac{x^2}{\sqrt{9-x^2}}dx = \int_0^{\pi/2} \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} 3\cos\theta d\theta = 9\int_0^{\pi/2} \sin^2\theta d\theta$$

$$= \frac{9}{2}\int_0^{\pi/2} (1-\cos 2\theta) d\theta = \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = ?$$

$\boxed{6}$ $\int \frac{\sqrt{1+t^2}}{t}dt$

$t = \tan\theta \Rightarrow dt = \sec^2\theta d\theta \quad , \; \theta = \tan^{-1} t$

$\int \frac{\sqrt{1+t^2}}{t}dt = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta d\theta = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta$

$= \int \frac{1}{\cancel{\cos\theta}} \frac{\cancel{\cos\theta}}{\sin\theta}\sec^2\theta d\theta = \int \csc\theta \sec^2\theta d\theta \qquad (by \;\; part)$

$u = \csc\theta \qquad dv = \sec^2\theta d\theta$

$du = -\csc\theta\cot\theta \qquad v = \tan\theta d\theta$

$= \tan\theta\csc\theta + \int \csc\theta d\theta = \tan\theta\csc\theta + Ln\left|\csc\theta + \cot\theta\right|$

$= \tan\tan^{-1} t \csc\tan^{-1} t + Ln\left|\csc\tan^{-1} t + \cot\tan^{-1} t\right| + C$

$\boxed{7}$ $\int \frac{dx}{\sqrt{x^2-1}}$

$x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta d\theta \quad , \theta = \sec^{-1}(x)$

$= \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\sec^2\theta - 1}} = \quad = \int \frac{\sec\theta\tan\theta d\theta}{\tan\theta} = \int \sec\theta d\theta$

$= Ln\left|\sec\theta + \tan\theta\right| = Ln\left|\sec\sec^{-1}(x) + \tan\sec^{-1}(x)\right|$

$= Ln\left|x + \tan\sec^{-1}(x)\right| + C$

$$\boxed{\int \sin^n x\,dx = \frac{-1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx}$$

EXAM :

$$\int \sin^3 x\,dx = \frac{-1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\,dx$$

$$= \frac{-1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x + C$$

$$\int \sin^4 x\,dx = \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx$$

$$= \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\left(\frac{-1}{2}\sin x\cos x + \frac{1}{2}\int dx\right)$$

$$= \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\left(\frac{-1}{2}\sin x\cos x + \frac{1}{2}x\right) + C$$

$$\boxed{\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx}$$

EXAM :

$$\int \cos^3 x\,dx = \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\int \cos x\,dx$$

$$= \frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C$$

$$\boxed{\begin{aligned}\int \sin^3 x\,dx &= \frac{1}{3}\cos^3 x - \cos x\\ \int \cos^3 x\,dx &= \sin x - \frac{1}{3}\sin^3 x\end{aligned}} \Leftarrow \textit{special rules}$$

If m & n are positive number (inteager) then : -

$$\int \sin^m x \cos^n x \, dx$$

$$n \quad odd \quad \Rightarrow\Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$n \quad even \quad \Rightarrow\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$m, n \quad odd \quad \Rightarrow\Rightarrow \cos^2 x = 1 - \sin^2 x \quad or \quad \sin^2 x = 1 - \cos^2 x$$

$$m, n \quad even \quad \Rightarrow\Rightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad and \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

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ملاحظات مهمة :

1- اذا اعطى في السؤال اس الـ Sin فردي و اس الـ Cos زوجي فيجب تحويل كل $Sin^2x=1-Cos^2x$

2- اذا اعطى في السؤال اس الـ Cos فردي و اس الـ Sin زوجي فيجب تحويل كل $Cos^2x=1-Sin^2x$

3- اذا اعطى الاسان فرديان فيجب تحويل احدهما اما $Sin^2x=1-Cos^2x$ او $Cos^2x=1-Sin^2x$

وللسهولة يجب ان نختار الدالة ذات الاس الاقل .

4 – اما اذا كان الاسان زوجيان فيجب تحويل الدالتين حسب الصيغ التالية

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad and \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

EXAM :

$$\int \sin^2 x \cos^3 x dx$$

$$= \int \sin^2 x \cos^2 x \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$$= \int (\sin^2 x \cos x - \sin^4 x \cos x) \, dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

$$\int \cos^2 x \sin^3 x \, dx$$

$$= \int \cos^2 x \sin^2 x \sin x \, dx = \int \cos^2 x (1 - \cos^2 x) \sin x \, dx$$

$$= \int (\cos^2 x \sin x - \cos^4 x \sin x) \, dx = \frac{-1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

$$\int \sin^5 x \cos^5 x \; dx$$

$$= \int (\sin^2 x)^2 \sin x \cos^5 x \; dx = \int (1-\cos^2 x)^2 \sin x \cos^5 x \; dx$$

$$= \int (1-2\cos^2 x + \cos^4 x) \sin x \cos^5 x \; dx$$

$$= \int (\sin x \cos^5 x - 2\sin x \cos^7 x + \sin x \cos^9 x)\, dx$$

$$= \frac{-1}{6}\cos^6 x + \frac{1}{4}\cos^8 x - \frac{1}{10}\cos^{10} x + C$$

$$\int \sin^2 x \cos^2 x \; dx$$

$$= \int \frac{1}{2}(1-\cos 2x).\frac{1}{2}(1+\cos 2x)\, dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\, dx = \frac{1}{4}\int \sin^2 2x \; dx$$

$$= \frac{1}{4}\frac{1}{2}\int (1-\cos 4x)\, dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + C$$

$$\boxed{\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx}$$

EXAM :

$$\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x - \int \tan x\, dx$$

$$= \frac{1}{2}\tan^2 x + Ln\left|\cos x\right| + C$$

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\, dx$$

$$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

$$\boxed{\int \sec^n x\, dx = \frac{1}{n-1}\sec^{n-2} x\, \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx}$$

EXAM :

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x\, \tan x + \frac{1}{2}\int \sec x\, dx$$

$$= \frac{1}{2}\sec x\, \tan x + \frac{1}{2}Ln\left|\sec x + \tan x\right| + C$$

$$\int \sec^4 x\, dx = \frac{1}{3}\sec^2 x\, \tan x + \frac{2}{3}\int \sec^2 x\, dx$$

$$= \frac{1}{3}\sec^2 x\, \tan x + \frac{2}{3}\tan x + C$$

ملاحظات مهمة :

1- اذا اعطى في السؤال اس الـ Sec زوجي فيجب تحويل $Sec^2=1+tan^2$ ومن ثم حلها باستخدام قاعدة $tan^n x$

أو :

اذا اعطى في السؤال اس الـ Sec زوجي فيجب تحويل $Sec^2=1+tan^2$ مع ترك Sec^2 واحدة لغرض المشتقة والتكامل مباشر .

2- اذا اعطى في السؤال اس الـ Sec فردي و أس الـ tan زوجي فيجب تحويل $tan^2=Sec^2-1$ ومن ثم حلها باستخدام قاعده $Sec^n x$

3- اذا اعطى اس ال Sec , tan فرديان فيجب تحويل $tan^2=Sec^2-1$ ونقوم بسحب tan واحدة و Sec واحدة لغرض المشتقة ومن ثم تكاملها مباشرة .

$$\int \tan^m x \sec^n x \, dx$$

$$n \ \ even \qquad \Rightarrow\Rightarrow \quad \sec^2 x = 1 + \tan^2 x$$

$$m \ \ odd \qquad \Rightarrow\Rightarrow \quad \tan^2 x = \sec^2 x - 1$$

$$m \ \ even, n \ odd \quad \Rightarrow\Rightarrow \quad \tan^2 x = \sec^2 x - 1$$

EXAM :

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \, \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + \tan^2 x) \sec^2 x \, dx = \int (\tan^3 x \sec^2 x + \tan^5 x \sec^2 x) \, dx$$

$$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \, \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + \tan^2 x) \sec^2 x \, dx = \int (\tan^3 x \sec^2 x + \tan^5 x \sec^2 x) \, dx$$

$$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$$

$$\int \tan^4 x \sec^3 x \, dx = \int (\sec^2 x - 1)^2 \sec^3 x \, dx$$

$$= \int (\sec^4 x - 2\sec^2 x + 1) \sec^3 x \, dx = \int (\sec^7 x - 2\sec^5 x + \sec^3 x) dx$$

$$\because \int \sec^7 x \, dx = \frac{1}{6} \sec^5 x \tan x + \frac{5}{6} \int \sec^5 x \, dx$$

$$\int \tan^4 x \sec^3 x \, dx = \frac{1}{6}\sec^5 x \tan x + \frac{5}{6}\int \sec^5 x \, dx - 2\int \sec^5 x \, dx + \int \sec^3 x dx$$

$$= \frac{1}{6}\sec^5 x \tan x - \frac{7}{6}\int \sec^5 x \, dx + \int \sec^3 x dx$$

$$= \frac{1}{6}\sec^5 x \tan x - \frac{7}{6}\left(\frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x dx\right) + \int \sec^3 x dx$$

$$= \frac{1}{6}\sec^5 x \tan x - \frac{7}{24}\sec^3 x \tan x + \frac{1}{8}\int \sec^3 x dx$$

$$= \frac{1}{6}\sec^5 x \tan x - \frac{7}{24}\sec^3 x \tan x + \frac{1}{8}\left(\frac{1}{2}\sec x \tan x + \frac{1}{2}Ln\left|\sec x + \tan x\right|\right) + C$$

$$\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \, (\sec^2 x - 1).\sec x \tan x \, dx$$

$$= \int (\sec^4 x .\sec x \tan x - \sec^2 x .\sec x \tan x)\, dx =$$

$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

$$\int \sec^4 x \tan^4 x \, dx = \int \sec^2 x \, (1 + \tan^2 x)\tan^4 x \, dx$$

$$= \int (\sec^2 x \tan^4 x + \sec^2 x \tan^6 x)\, dx =$$

$$= \frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C$$

$$\boxed{\begin{aligned}\sin A\cos B &= \frac{1}{2}\left[\sin(A-B)+\sin(A+B)\right]\\ \sin A\sin B &= \frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]\\ \cos A\cos B &= \frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]\end{aligned}}$$

EXAM :

$$\int \sin 5x\cos 3x\,dx = \frac{1}{2}\int(\sin 2x+\sin 8x)\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2x-\frac{1}{8}\cos 8x\right]+C$$

$$\int \sin 4x\sin x\,dx = \frac{1}{2}\int(\cos 3x-\cos 5x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\sin 3x-\frac{1}{5}\sin 5x\right)+C$$

$$\int \cos 7x\cos 2x\,dx = \frac{1}{2}\int(\cos 5x+\cos 9x)\,dx$$

$$= \frac{1}{2}\left(\frac{1}{5}\sin 5x+\frac{1}{9}\sin 9x\right)+C$$