DEFINITE INTEGRAL APPLICATIONS

1 Area

a) Area between f(x) and the axis

$$A=\int_a^b f(x)dx=\int_a^b y\,dx \qquad , \quad area\ respected\ to\ x-axis$$

$$A=\int_c^d g(y)dy=\int_a^b x\,dy \qquad , \quad area\ respected\ to\ y-axis$$

b) Area between two curves

$y_1=f(x_1)$, $y_2=f(x_2)$, $x_1=g(y_1)$, $x_2=g(y_2)$

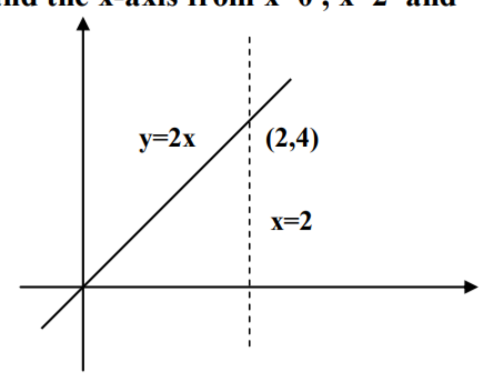
$$A=\int_a^b (y_1-y_2)dx \quad ,x-axis \quad ,y_1\ge y_2$$

$$A=\int_c^d (x_1-x_2)dy \quad ,y-axis \quad ,x_1\ge x_2$$

ملاحظات هامة :

1) اذا اعطانا دالة وحدود تكامل فالحل يكون مباشر .
2) اذا اعطانا دالتين بدون حدود تكامل فنقاطع الدالتين .
3) اذا اعطانا دالة فقط بدون حدود تكامل ، فالحل يكون بالاعتماد على المحور ، فاذا اعطى المساحة بالنسبة للمحور x نضيف معادلة y=0 واذا اعطى المساحة بالنسبة للمحور y نضيف معادلة x=0 ، ثم نقاطع الدالتين .
4) اذا طلب المساحة بالنسبة للمحور x وحدود التكامل معطاة بالنسبة للمحور y فيجب تحويل حدود التكامل بدلالة x والعكس صحيح .

<u>EXAM</u> :- Find the area bounded by the line y=2x and the x-axis from x=0 , x=2 and check the result by geometrically.



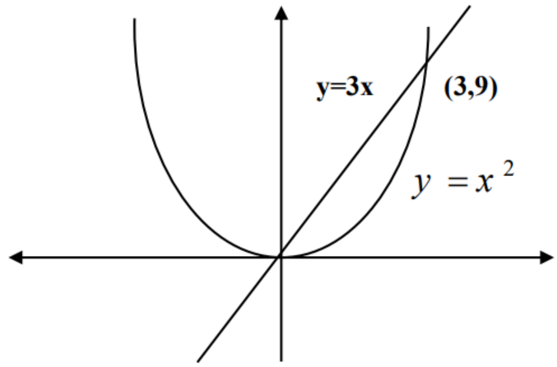
$$A=\int_a^b y\,dx=\int_0^2 2x\,dx=x^2\Big|_0^2=4\,unit^2$$

In Geometry

$$_{\triangle}A=\frac{1}{2}b.h=\frac{1}{2}(2)(4)=4$$

<u>EXAM</u> :- Find the area between the curves :

$y = x^2$ *and the line* $y = 3x$



Solution :

$y = x^2$ ....(1)

$y = 3x$ ....(2)

$x^2 = 3x \Rightarrow x^2 - 3x = 0$

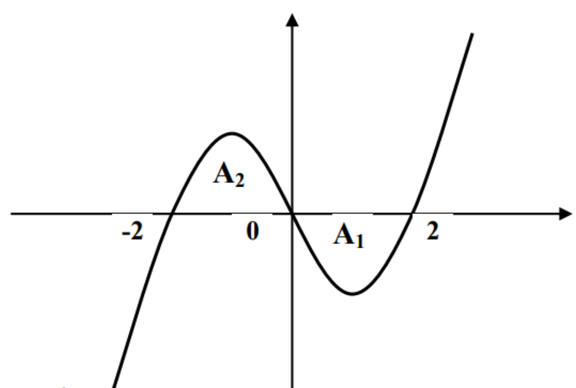
$x(x-3) = 0 \Rightarrow x = 0, x = 3$

$$A = \int_a^b (y_1 - y_2)dx \quad , y_1 \geq y_2$$

$$= \int_0^3 (3x - x^2)dx = \frac{3}{2}x^2 - \frac{1}{3}x^3 \Big|_0^3 = 4.5\, unit^2$$

<u>EXAM</u> :- Find the area between the curves :

$y = x(x^2 - 4)$ **and the x-axis**



Solution :

$y = x(x^2 - 4)$

$y = 0$

$\therefore x(x^2 - 4) = 0 \Rightarrow x = 0, x = -2, x = 2$

$A = |A_1| + |A_2|$

$$A_1 = \int_{-2}^{0} x(x^2 - 4)dx = \int_{-2}^{0} (x^3 - 4x)dx = \frac{1}{4}x^4 - 2x^2 \Big|_{-2}^{0} = 4$$

$$A_2 = \int_{0}^{2} x(x^2 - 4)dx = \int_{0}^{2} (x^3 - 4x)dx = \frac{1}{4}x^4 - 2x^2 \Big|_{0}^{2} = -4$$

$A = |4| + |-4| = 8$

EXAM :- Find the area between the line x-y=10 and the

a) x-axis b)y-axis

from x=0 , x=5

$$a)\quad A=\int_a^b y\,dx \qquad , \quad x-y=10 \Rightarrow y=x-10$$

$$=\int_0^5 (x-10)dx=\frac{x^2}{2}-10x\Big|_0^5=|-37.5|=37.5$$

$$b)\quad A=\int_c^d x\,dy \qquad , \quad x-y=10 \Rightarrow x=y+10$$

$$A=\int_{-10}^{-5}(y+10)dy \qquad x=0\rightarrow y=-10 \quad \& x=5\rightarrow y=-5$$

$$=\frac{1}{2}y^2+10y\Big|_{-10}^{-5}=?$$

2 Volumes

DEF: if f(x) ≥ 0 cont. on [a,b] if the area bounded by f(x) and the x-axis from x=a to x=b rotated about the x-axis then :

$$V_x=\int_a^b \pi[f(x)]^2dx=\int_a^b \pi y^2dx$$

DEF: if g(y) ≥ 0 cont. on [c,d] if the area bounded by g(y) and the y-axis fromy=c to y=d rotated about the y-axis then :

$$V_y=\int_c^d \pi[g(y)]^2dy=\int_c^d \pi x^2dy$$

DEF : if the area bounded between two curves is rotated about x-axis

$$V_x=\int_a^b \pi(y_1^2-y_2^2)dx \quad ,y_1\geq y_2$$

DEF : if the area bounded between two curves is rotated about y-axis

$$V_y=\int_c^d \pi(x_1^2-x_2^2)dy \quad ,x_1\geq x_2$$

<u>EXAM</u> :- The area bounded by the function $x^2y^2=1$ is rotated about the y-axis, find the volume generated from y=1 , y=2 .

$$V_y=\pi\int_c^d x^2dy \qquad , \quad x^2y^2=1\rightarrow x^2=y^{-2}$$

$$=\pi\int_{.c}^{d} y^{-2}dy=\left.\frac{-1}{y}\right|_1^2=\frac{1}{2}\pi\,unit^3$$

<u>EXAM</u> :- The area bounded by the line y=x-1
a) is rotated about the x-axis b) is rotated about the y-axis
find the volume generated from those rotation from x=0,1

$$a)V_x=\pi\int_a^b y^2dx \qquad , \quad y=x-1\Rightarrow y^2=(x-1)^2$$

$$=\pi\int_0^1(x-1)^2dx=\left.\frac{\pi}{3}(x-1)^3\right|_0^1=\frac{\pi}{3}unit^3$$

$$b)V_y=\pi\int_c^d x^2dy \qquad , \quad y=x-1\Rightarrow x=y+1\Rightarrow x^2=(y+1)^2$$

$$x=0\rightarrow y=-1 \quad \& x=1\rightarrow y=0$$

$$=\pi\int_{-1}^{0}(y+1)^2dx=\left.\frac{\pi}{3}(y+1)^3\right|_{-1}^{0}=\frac{\pi}{3}unit^3$$

3 Arc Length

If y=f(x) is continous with continous derivative at each point of the curve from (a,f(a)) to (b,f(b)) then :

$$S=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}dx \qquad ,if \quad \frac{dy}{dx}\ is\ cont.$$

$$S=\int_c^d\sqrt{1+(\frac{dx}{dy})^2}dy \qquad ,if \quad \frac{dy}{dx}\ is\ not\ cont.$$

<u>EXAM</u> :- Find the length of the segment of curve $y=\frac{1}{3}(x^2+2)^{3/2} \quad ,from\ x=0,3$

$$\frac{dy}{dx}=\frac{1}{3}\frac{3}{2}(x^2+2)^{1/2}(2x)=x(x^2+2)^{1/2}\ cont.on[0,3]$$

$$S=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}dx=\int_0^3\sqrt{1+(x(x^2+2)^{1/2})^2}dx$$

$$=\int_0^3\sqrt{1+x^2(x^2+2)}dx=\int_0^3\sqrt{x^4+2x^2+1}dx$$

$$=\int_0^3\sqrt{(x^2+1)^2}dx=\int_0^3(x^2+1)dx$$

$$=\frac{x^3}{3}+x\Big|_0^3=?$$

Home work

1- Find the area bounded by the line 2x-5y=10 from x=0 , 10 about the x&y – axis.
2- Find the area bounded by the curves :

$y=x^2 \quad , \quad y=\sqrt{x} \quad ,x=1,2 \quad ,x-axis$

$y=\sec^2 x \quad , \quad y=x \quad ,x=-45,45 \quad ,x-axis$

$y=x^2=y \quad , \quad x=y-2 \quad ,x-axis$

$y=x^3-2x^2 \quad , \quad y=2x^2-3x \quad ,x-axis$

3- Find the volume generated from rotation between curves :

$y=\sec x \quad , \quad y=0 \quad ,x=45,60 \quad ,x-axis$

$y=\sqrt{25-x^2} \quad , \quad y=3 \quad ,x-axis$

$y=c\,sc x \quad , \quad y=2 \quad ,x=45,60 \quad ,x-axis$

$x=1-y^2 \quad , \quad x=2+y^2 \quad ,y=-1,1 \quad ,y-axis$

$y=\sin x \quad , \quad y=\cos x \quad ,x=0,45 \quad ,x-axis$

$y=\tan x \quad , \quad y=-1 \quad ,x=45,60 \quad ,x-axis$