In Mathematics, the hyperbolic functions are similar to the trigonometric functions or circular functions. Generally, the hyperbolic functions are defined through the algebraic expressions that include the exponential function (e^x) and its inverse exponential functions (e^{-x}), where e is the Euler's constant.

في الرياضيات، الدوال الزائدية هي نظائر للدوال المثلثية العادية، ولكن يتم تعريفها باستخدام القطع الزائد بدلاً من الدائرة. مثلما تشكل النقاط (cos t، sin t) دائرة ذات نصف قطر وحدة، فإن النقاط (cosh t، sinh t) تشكل النصف الأيمن من وحدة القطع الزائد

TRIGONOMETRY HYPERBOLIC FUNCTIONS الدوال الزائدية

$$\boxed{1}\ Sinh(x) = \frac{e^x - e^{-x}}{2} \quad , D_f = \mathbb{R} \ , \ R_f = \mathbb{R}$$

$$\boxed{2}\ Cosh(x) = \frac{e^x + e^{-x}}{2} \quad , D_f = \mathbb{R} \ , \ R_f = [1,\infty)$$

$$\boxed{3}\ \tan h(x) = \frac{Sinh(x)}{Cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \ , D_f = \mathbb{R} \ , \ R_f = (-1,1)$$

$$\boxed{4}\ \cot h(x) = \frac{Cosh(x)}{Sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} , D_f = \mathbb{R} - \{0\} \ , \ R_f = \mathbb{R} - \{0\}$$

$$\boxed{5}\ Sech(x) = \frac{1}{Cosh(x)} = \frac{2}{e^x + e^{-x}} \ , D_f = \mathbb{R} \ , \ R_f = (0,1]$$

$$\boxed{6}\ \csc h(x) = \frac{1}{Sinh(x)} = \frac{2}{e^x - e^{-x}} , D_f = \mathbb{R} - \{0\} \ , \ R_f = \mathbb{R} - [-1,1]$$

Some rules :

$\langle 1 \rangle$ $\cosh(x)+\sinh(x)=e^{x}$ $\qquad$ $\langle 2 \rangle$ $\cosh(x)-\sinh(x)=e^{-x}$

$\langle 3 \rangle$ $\cosh^2(x)-\sinh^2(x)=1$ $\quad$ *(main rule)*

$\langle 4 \rangle$ $1-\tanh^2(x)=\operatorname{sec}h^2(x)$ $\qquad$ $\langle 5 \rangle$ $\coth^2(x)-1=\operatorname{csc}h^2(x)$

$\langle 6 \rangle$ $\cosh^2(x)=\frac{1}{2}(\cosh 2x+1)$ $\qquad$ $\langle 7 \rangle$ $\sinh^2(x)=\frac{1}{2}(\cosh 2x-1)$

$\langle 8 \rangle$ $\sinh(x\pm y)=\sinh x\cosh y\pm\sinh y\cosh x \Rightarrow \sinh(2x)=2\sinh x\cosh y$

$\langle 9 \rangle$ $\cosh(x\pm y)=\cosh x\cosh y\pm\sinh x\sinh y \Rightarrow \cosh(2x)=\cosh^2 x+\sinh^2 x$

$\langle 10 \rangle$ $\tanh(x\pm y)=\dfrac{\tanh x\pm\tanh y}{1\pm\tanh x\cdot\tanh y} \Rightarrow \tanh(2x)=\dfrac{2\tanh x}{1+\tanh^2 x}$

The proof:

$$1)\ \cosh(x)+\sinh(x)=\frac{e^{x}+e^{-x}}{2}+\frac{e^{x}-e^{-x}}{2}$$

$$=\frac{2e^{x}}{2}=e^{x}$$

$$2)\ \cosh(x)-\sinh(x)=\frac{e^{x}+e^{-x}}{2}-\frac{e^{x}-e^{-x}}{2}$$

$$=\frac{2e^{-x}}{2}=e^{-x}$$

$$3)\ \cosh^2(x)-\sinh^2(x)=[\cosh(x)+\sinh(x)][\cosh(x)-\sinh(x)]$$

$$=e^{x}\cdot e^{-x}=e^{0}=1$$

Ex: Find the value of cosh(0) , sinh(0)

$$\cosh(0)=\frac{e^{0}+e^{-0}}{2}=1$$

$$\sinh(0)=\frac{e^{0}-e^{-0}}{2}=0$$

HYPERBOLIC TRIG. FUN. DERIVATIVE :

$\langle 1\rangle\ \frac{d}{dx}\sinh x=\cosh x \qquad \Rightarrow Gen.\Rightarrow \frac{d}{dx}\sinh u=\cosh u\cdot du$

$\langle 2\rangle\ \frac{d}{dx}\cosh x=\sinh x \qquad \Rightarrow Gen.\Rightarrow \frac{d}{dx}\cosh u=\sinh u\cdot du$

$\langle 3\rangle\ \frac{d}{dx}\tanh x=\operatorname{sech}^2 x \qquad \Rightarrow Gen.\Rightarrow \frac{d}{dx}\tanh u=\operatorname{sech}^2 u\cdot du$

$\langle 4\rangle\ \frac{d}{dx}\coth x=-\operatorname{csch}^2 x \qquad \Rightarrow Gen.\Rightarrow \frac{d}{dx}\coth u=-\operatorname{csch}^2 u\cdot du$

$\langle 5\rangle\ \frac{d}{dx}\operatorname{sech} x=-\sec hx\ \tanh x \Rightarrow Gen.\Rightarrow \frac{d}{dx}\operatorname{sech} u=-\sec hu\ \tanh u\cdot du$

$\langle 6\rangle\ \frac{d}{dx}\operatorname{csch} x=-\csc hx\ \coth x \Rightarrow Gen.\Rightarrow \frac{d}{dx}\operatorname{csch} u=-\csc hu\ \coth u\cdot du$

PROOF:

$$\langle 1\rangle\ \frac{d}{dx}\sinh x=\frac{d}{dx}\frac{e^x-e^{-x}}{2}=\frac{e^x+e^{-x}}{2}=\cosh x$$

$$\langle 3\rangle\ \frac{d}{dx}\tanh x=\frac{d}{dx}[\frac{e^x-e^{-x}}{e^x+e^{-x}}]=\frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$=\frac{(e^{2x}+2+e^{-2x})-(e^{2x}+2+e^{-2x})}{(e^x+e^{-x})^2}$$

$$=\frac{4}{(e^x+e^{-x})^2}=[\frac{2}{e^x+e^{-x}}]^2$$

$$=\sec h^2 x$$

<u>Exam</u> :Find $\frac{dy}{dx}$ to the functions :

$\boxed{1}\ y=\tanh(x^2)$

$y'=2x\sec h^2(x^2)$

$\boxed{2}$ $y = e^{\sinh 3x}$

$$y' = e^{\sinh 3x} \cdot \cosh 3x \cdot 3$$

$\boxed{3}$ $y = Ln(1+\cosh 4x)$

$$y' = \frac{4\sinh 4x}{1+\cosh 4x}$$

$\boxed{4}$ $y = (\csc hx)^{Lnx}$

$$Lny = Ln(\csc hx)^{Lnx} = Lnx \cdot Ln(\csc hx)$$

$$\frac{1}{y}y' = Lnx \cdot \frac{-\csc hx\ \coth x}{\csc hx} + \frac{Ln(\csc hx)}{x}$$

$$y' = y[Lnx \cdot -\coth x + \frac{Ln(\csc hx)}{x}]$$

$$y' = (\csc hx)^{Lnx}[-\coth x \cdot Lnx + \frac{Ln(\csc hx)}{x}]$$

INVERSE HYPERBOLIC TRIG. FUN.

$\langle 1\rangle$ $\sinh^{-1}(x) = Ln(x + \sqrt{x^2+1})$

$\langle 2\rangle$ $\cosh^{-1}(x) = Ln(x + \sqrt{x^2-1})$

$\langle 3\rangle$ $\tanh^{-1}(x) = \frac{1}{2}Ln(\frac{1+x}{1-x})$

$\langle 4\rangle$ $\coth^{-1}(x) = \frac{1}{2}Ln(\frac{x+1}{x-1})$

$\langle 5\rangle$ $\text{sech}^{-1}(x) = Ln(\frac{1+\sqrt{1-x^2}}{x})$

$\langle 6\rangle$ $cs\,\text{ch}^{-1}(x) = Ln(\frac{1+\sqrt{1+x^2}}{x})$

INVERSE HYPERBOLIC TRIG. FUN. DERIVATIVE :

$\langle 1\rangle \; \frac{d}{dx}\sinh^{-1}(u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

$\langle 2\rangle \; \frac{d}{dx}\cosh^{-1}(u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

$\langle 3\rangle \; \frac{d}{dx}\tanh^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx}$

$\langle 4\rangle \; \frac{d}{dx}\coth^{-1}(u) = \frac{1}{1-u^2}\frac{du}{dx}$

$\langle 5\rangle \; \frac{d}{dx}\operatorname{sech}^{-1}(u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$

$\langle 6\rangle \; \frac{d}{dx}cs\,\mathrm{ch}^{-1}(u) = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$

exam :

1) $y = \sinh^{-1}(3x)$

$$y' = \frac{3}{\sqrt{1+9x^2}}$$

2) $y = \cosh^{-1}(e^x)$

$$y' = \frac{e^x}{\sqrt{e^{2x}-1}}$$

3) $y = \operatorname{sech}^{-1}(\ln x)$

$$y' = \frac{-1}{x\ \ln x\sqrt{1-\ln^2 x}}$$

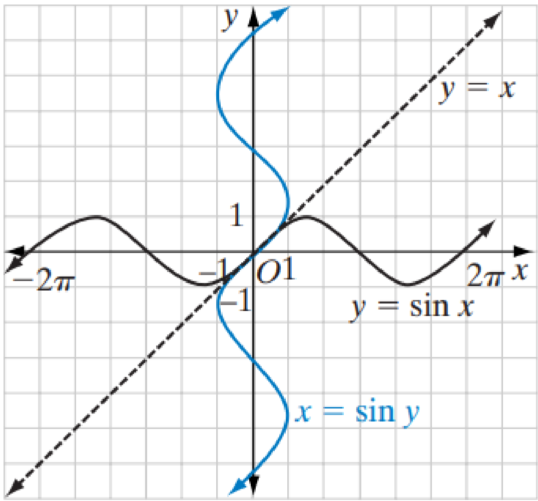
4) $y = (\coth^{-1}(e^{-x}))^5$

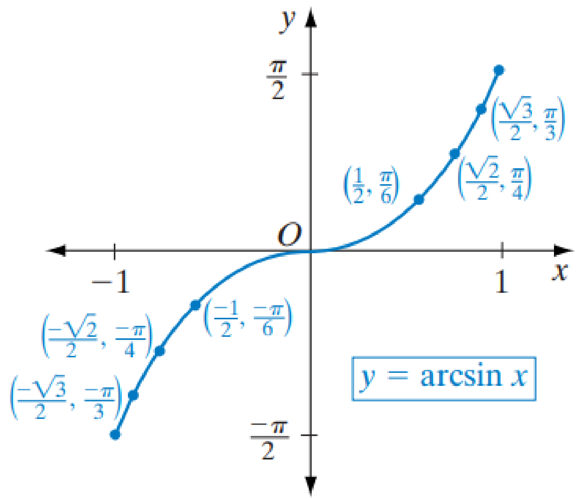
$$y' = 5(\coth^{-1}(e^{-x}))^4[\frac{-e^{-x}}{1-e^{-2x}}]$$

Homework

$\boxed{1}$ *if* $\tanh(x) = \frac{-4}{5}$, *show that* : $\sinh(x)+\cosh(x) = \frac{1}{3}$

$\boxed{2}$ *Find the value of* $\cosh(\ln 4)$?





Sine Function with a Restricted Domain
$y = \sin x$
Domain $= \left\{x : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$
Range $= \{y : -1 \leq y \leq 1\}$

Inverse Sine Function
$y = \arcsin x$
Domain $= \{x : -1 \leq x \leq 1\}$
Range $= \left\{y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

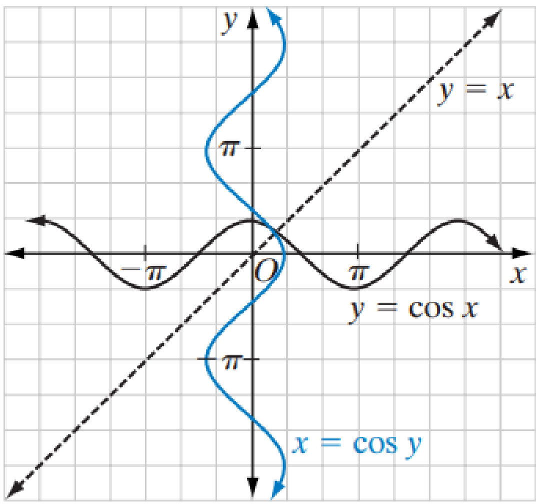
The trigonometric functions are not one-to-one functions. By restricting the domain of each function

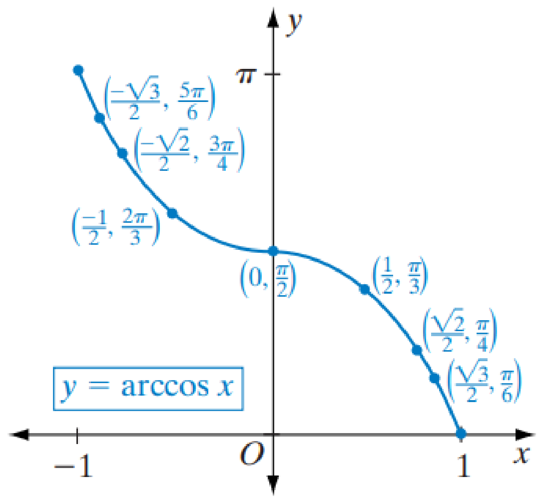
Inverse of the Sine Function

The graph of $y = \sin x$ is shown below on the left. When the graph of $y = \sin x$ is reflected in the line $y = x$, the image, $x = \sin y$ or $y = \arcsin x$, is a relation that is not a function. The interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ includes all of the values of $\sin x$ from -1 to 1.

Inverse of the Cosine Function

When the graph of $y = \cos x$ is reflected in the line $y = x$, the image, $x = \cos y$ or $y = \arccos x$, is a relation that is not a function. The interval $0 \le x \le \pi$ includes all of the values of $\cos x$ from -1 to 1.



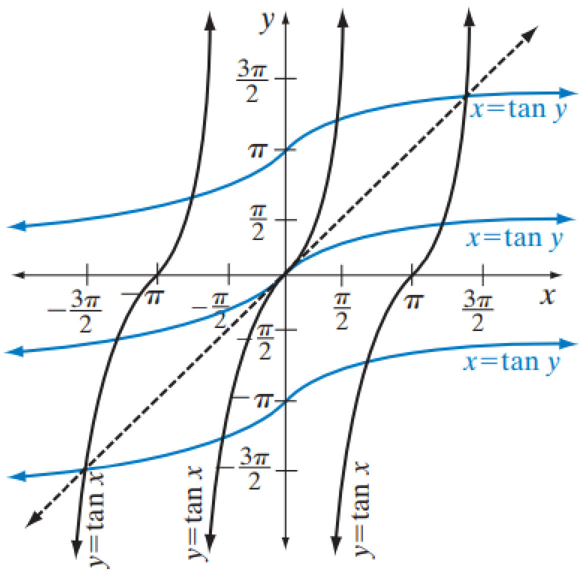


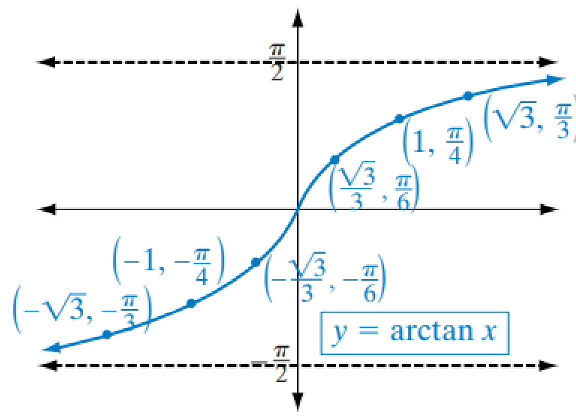
Cosine Function with a Restricted Domain
$y = \cos x$
Domain $= \{x : 0 \le x \le \pi\}$
Range $= \{y : -1 \le y \le 1\}$

Inverse Cosine Function
$y = \arccos x$
Domain $= \{x : -1 \le x \le 1\}$
Range $= \{y : 0 \le y \le \pi\}$

Inverse of the Tangent Function

The graph of $y = \tan x$ is shown below on the left. When the graph of $y = \tan y$ is reflected in the line $y = x$, the image, $x = \tan y$ or $y = \arctan x$, is a relation that is not a function. The interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ includes all of the values of tan x from $-\infty$ to ∞.





Tangent Function with a Restricted Domain	**Inverse Tangent Function**
$y = \tan x$	**$y = \arctan x$**
Domain $= \left\{x : -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$	Domain $= \{x : x$ is a real number$\}$
Range $= \{y : y$ is a real number$\}$	Range $= \left\{y : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$