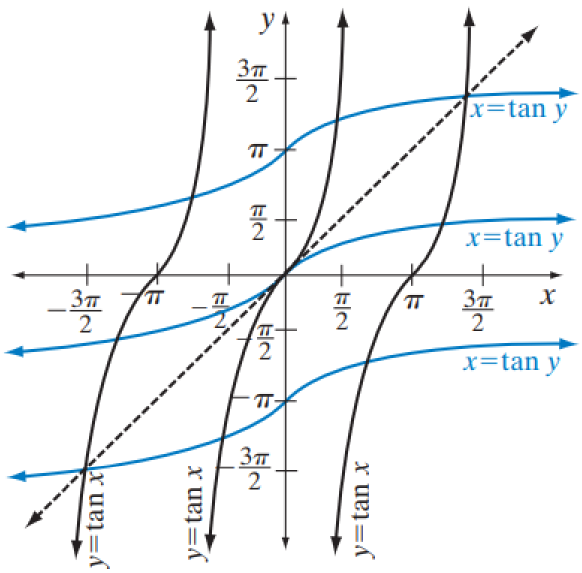
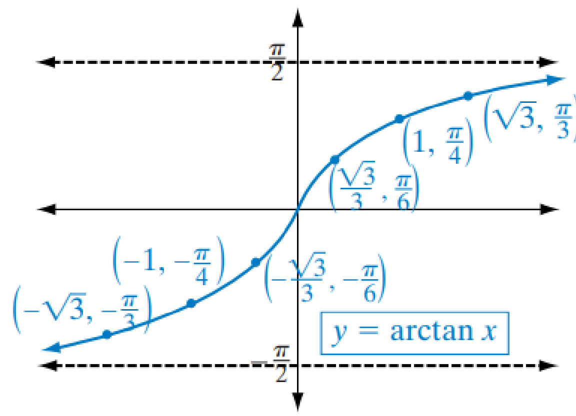
Inverse of the Tangent Function

The graph of $y = \tan x$ is shown below on the left. When the graph of $y = \tan y$ is reflected in the line $y = x$, the image, $x = \tan y$ or $y = \arctan x$, is a relation that is not a function. The interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ includes all of the values of tan x from $-\infty$ to ∞.

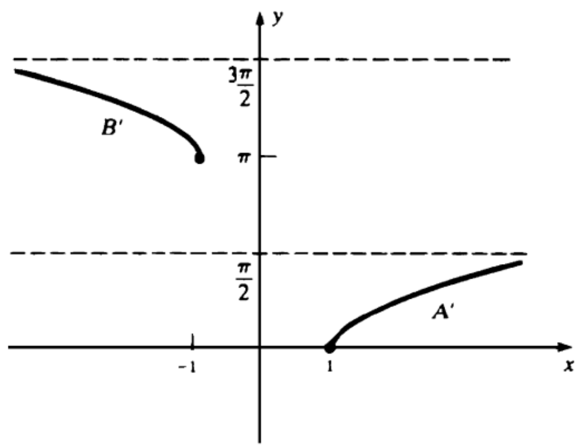




Tangent Function with a Restricted Domain	Inverse Tangent Function
y = tan x	**y = arctan x**
Domain $= \left\{x : -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$	Domain $= \{x : x$ is a real number$\}$
Range $= \{y : y$ is a real number$\}$	Range $= \left\{y : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

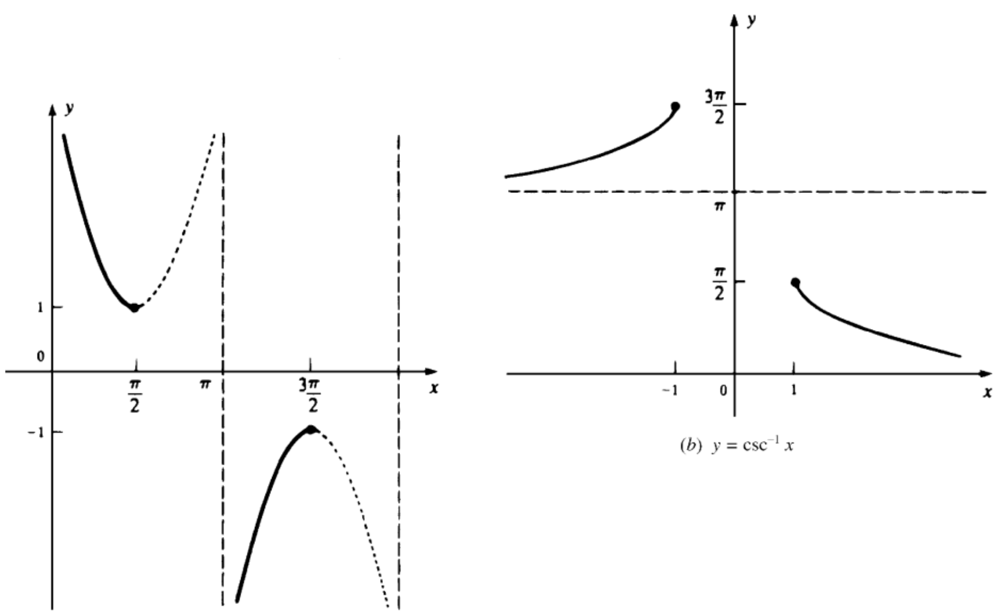
$\sec^{-1} x$. Restrict $\sec x$ to the union of $[0, \pi/2)$ and $[\pi, 3\pi/2)$. Then the domain of $\sec^{-1} x$ consists of all y such that $|y| \geq 1$ and

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x$$



$y = \sec^{-1} x$

$\csc^{-1} x$. Restrict $\csc x$ to the union of $(0, \pi/2]$ and $(\pi, 3\pi/2]$. Then the domain of $\csc^{-1} x$ consists of all y such that $|y| \geq 1$ and



(b) $y = \csc^{-1} x$

(a) $y = \csc x$

The above definition implies that the signs of $\cos\theta$ and $\sin\theta$ are determined by the quadrant in which point B lies. In the first quadrant, $\cos\theta > 0$ and $\sin\theta > 0$. In the second quadrant, $\cos\theta < 0$ and $\sin\theta > 0$. In the third quadrant, $\cos\theta < 0$ and $\sin\theta < 0$. In the fourth quadrant, $\cos\theta > 0$ and $\sin\theta < 0$

Example

Differentiate $\log_e(x^2+3x+1)$.

Solution

$$\begin{aligned}\frac{d}{dx}\log_e(x^2+3x+1) &= \frac{1}{x^2+3x+1}\times\frac{d}{dx}(x^2+3x+1)\\ &= \frac{1}{x^2+3x+1}\times(2x+3)\\ &= \frac{2x+3}{x^2+3x+1}.\end{aligned}$$

Example Find $\frac{d}{dx}(e^{3x^2})$.

Solution

$$\begin{aligned}\frac{d}{dx}(e^{3x^2}) &= e^{3x^2}\times\frac{d}{dx}(3x^2)\\ &= 6xe^{3x^2}.\end{aligned}$$

$y = x\csc^{-1}\left(\frac{1}{x}\right) + \sqrt{1-x^2} \quad \text{for } 0 < x < 1.$

$$y' = x\left[\frac{1}{x}\frac{1}{\sqrt{\frac{1}{x^2}-1}}\right] + \csc^{-1}\left(\frac{1}{x}\right) + \tfrac{1}{2}(1-x^2)^{1/2}(-2x) = \csc^{-1}\left(\frac{1}{x}\right)$$

$y = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right).$

$$y' = \frac{1}{ab}\left[\frac{1}{1+\left(\frac{b}{a}\tan x\right)^2} D_x\left(\frac{b}{a}\tan x\right)\right] = \frac{1}{ab}\frac{a^2}{a^2+b^2\tan^2 x}\frac{b}{a}\sec^2 x = \frac{\sec^2 x}{a^2+b^2\tan^2 x}$$

$$= \frac{1}{a^2\cos^2 x + b^2\sin^2 x}$$

Find the derivative of the function.

a. $f(x) = \csc^{-1}(5x^2+1)$

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}\csc^{-1}(5x^2+1)\\ &= -\frac{1}{|5x^2+1|\sqrt{(5x^2+1)^2-1}}\frac{d}{dx}(5x^2+1)\\ &= -\frac{10x}{|5x^2+1|\sqrt{(5x^2+1)^2-1}}\end{aligned}$$

b. $f(x) = (\tan^{-1}(2x))^3$

$$f'(x) = \frac{6(\tan^{-1}(2x))^2}{1+4x^2}$$

c. $g(x)=\sqrt{e^{cos^{-1}(x)}}$

$$\begin{aligned}\frac{dg}{dx}&=\frac{d}{dx}\sqrt{e^{\cos^{-1}(x)}}\\&=\frac{1}{2\sqrt{e^{\cos^{-1}(x)}}}\frac{d}{dx}e^{\cos^{-1}(x)}\\&=\frac{1}{2\sqrt{e^{\cos^{-1}(x)}}}\cdot e^{\cos^{-1}(x)}\frac{d}{dx}\cos^{-1}(x)\\&=\frac{1}{2\sqrt{e^{\cos^{-1}(x)}}}\cdot e^{\cos^{-1}(x)}\cdot\frac{1}{\sqrt{1-x^2}}\\&=\frac{\sqrt{e^{\cos^{-1}(x)}}}{2\sqrt{1-x^2}}\end{aligned}$$

d. $f(t)=\ln(\sin^{-1}t)$

$$\begin{aligned}\frac{df}{dt}&=\frac{d}{dt}\ln\left(\sin^{-1}t\right)\\&=\frac{1}{\sin^{-1}t}\frac{d}{dt}\sin^{-1}(t)\\&=\frac{1}{\sin^{-1}t\sqrt{1-t^2}}\end{aligned}$$

e. $f(x)=\sec^{-1}(x^{3/2})$

$$f'(x)=\frac{2}{2|x|\sqrt{x^3-1}}$$

f. $h(s)=\cos^{-1}(\log_2 s)$

$$h'(s)=-\frac{1}{(\ln 2)s\sqrt{1-(\log_2 s)^2}}$$

g. $f(x) = (\cot^{-1} x)^{1/3}$

$$f'(x) = -\frac{1}{3(1+x^2)(\cot^{-1} x)^{2/3}}$$

h. $g(t) = \sin^{-1}(3^t)$

$$\begin{aligned}
\frac{dg}{dt} &= \frac{d}{dt}\sin^{-1}(3^t) \\
&= \frac{1}{\sqrt{1-(3^t)^2}}\,\frac{d}{dt}3^t \\
&= \frac{3^t \ln(3)}{\sqrt{1-3^{2t}}}
\end{aligned}$$

Given $f(x) = 1 + \ln(x-2)$, first show that f^{-1} exists, then compute $\left[f^{-1}\right]'(1)$.

Let $y = f(x) = 1 + \ln(x-2)$. Then,

$$\begin{aligned}
y - 1 &= \ln(x-2) \\
e^{y-2} + 2 &= x
\end{aligned}$$

Therefore $f^{-1}(x) = e^{x-1} + 2$.

$f^{-1}(1) = e^0 + 2 = 3$. So,

$$(f^{-1})'(1) = \left.\frac{1}{x-2}\right|_{x=3} = 1.$$

Find the derivatives of $\sin^{-1}(x) + \cos^{-1}(x)$ and $(x^2+1)\tan^{-1}(x)$.

We first differentiate $\sin^{-1} x + \cos^{-1} x$ using the formulas for the derivatives of inverse trigonometric functions:

$$\begin{aligned}\frac{d}{dx}\left(\sin^{-1} x + \cos^{-1} x\right) &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\ &= 0\end{aligned}$$

Next, we differentiate $(x^2+1)\tan^{-1} x$:

$$\begin{aligned}\frac{d}{dx}(x^2+1)\tan^{-1} x &= 2x\tan^{-1} x + \frac{x^2+1}{1+x^2} \\ &= 2x\tan^{-1} x + 1\end{aligned}$$

Differentiate $y = \sin^{-1}(x^2)$ and $y = \tan^{-1}(3x)$.

$$\frac{d}{dx}\tan^{-1}(3x) = \frac{3}{1+9x^2}$$

Example: Solve $\cosh^2 x - \sinh^2 x$

Solution:

Given: $\cosh^2 x - \sinh^2 x$

We know that

$\sinh x = [e^x - e^{-x}]/2$

$\cosh x = [e^x + e^{-x}]/2$

$\cosh^2 x - \sinh^2 x = [\ [e^x + e^{-x}]/2\ ]^2 - [\ [e^x - e^{-x}]/2\ ]^2$

$\cosh^2 x - \sinh^2 x = (4e^{x-x})\ /4$

$\cosh^2 x - \sinh^2 x = (4e^0)\ /4$

$\cosh^2 x - \sinh^2 x = 4(1)\ /4 = 1$

Therefore, $\cosh^2 x - \sinh^2 x = 1$