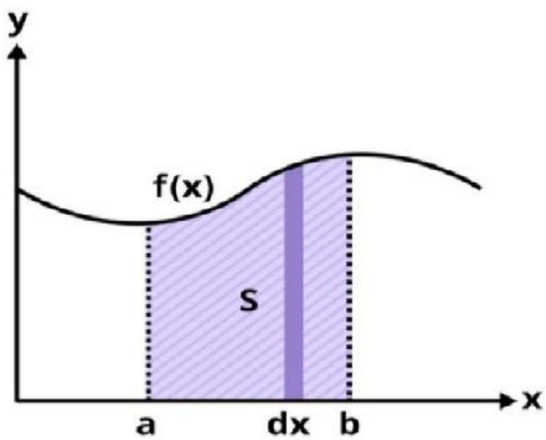
التكامل Integration

في علم الرياضيات ينقسم التكامل إلى جزأين: التكامل المحدود والتكامل غير المحدود .يتعلق التكامل المحدود بحساب الأطوال، المساحات، المنحنيات، مراكز الثقل وما إلى ذلك من الدوال التي لها تطبيقات في شتى العلوم .من جهة أخرى يركز التكامل غير المحدود على إيجاد المعكوس الرياضي للتفاضل، ولهذا السبب يسمى أيضًا بـالاشتقاق العكسي.

ويمكن تعريف التكامل هندسيًا على أنه مقدار المساحة تحت منحنى دالة ما



التكامل المحدد

التكامل المحدد (بالإنجليزية: Definite Integral)، هو التكامل الذي يكون فيه الحدان العلوي والسفلي معلومين

$$\int_1^3 (2x^2 + 3x + 2)\, dx$$

وتساوي نتيجة هذا التكامل نتيجة تعويض الحد العلوي في المعادلة الناتجة من التكامل مطروحًا من نتيجة تعويض الحد السفلي في المعادلة

التكامل غير المحدد

التكامل غير المحدد (بالإنجليزية: Indefinite Integral)، هو التكامل الذي يكون فيه الحدان العلوي والسفلي غير موجودين

$$\int (x^3 + 4x^2 + 10x)\, dx$$

Rules of Integration

1. The integral of a constant multiple of a function

A constant factor in an integral can be moved outside the integral sign in the following way.

$$\int k\, f(x)\,\mathrm{d}x = k \int f(x)\mathrm{d}x$$

Example

Find $\int 11x^2\,\mathrm{d}x$.

Solution

We are integrating a multiple of x^2. The constant factor,11, can be moved outside the integral sign.

$$\int 11x^2\,\mathrm{d}x = 11\int x^2\,\mathrm{d}x = 11\left(\frac{x^3}{3}+c\right) = \frac{11x^3}{3} + 11c$$

Example

Find $\int -5\cos x\,\mathrm{d}x$.

Solution

We are integrating a multiple of $\cos x$. The constant factor, -5, can be moved outside the integral sign.

$$\int -5\cos x\,\mathrm{d}x = -5\int \cos x\,\mathrm{d}x = -5\left(\sin x + c\right) = -5\sin x + K$$

where K is a constant.

2. The integral of the sum or difference of two functions

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

$$\int f(x) + g(x) \quad \mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$
$$\int f(x) - g(x) \quad \mathrm{d}x = \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x$$

Example
Find $\displaystyle\int (x^3 + \sin x)\mathrm{d}x$.

Solution

$$\int (x^3 + \sin x)\mathrm{d}x = \int x^3\,\mathrm{d}x + \int \sin x\,\mathrm{d}x = \frac{x^4}{4} - \cos x + c$$

Example
Find $\displaystyle\int \mathrm{e}^{3x} - x^7\,\mathrm{d}x$.

Solution

$$\int \mathrm{e}^{3x} - x^7\,\mathrm{d}x = \int \mathrm{e}^{3x}\,\mathrm{d}x \quad - \quad \int x^7\,\mathrm{d}x = \frac{\mathrm{e}^{3x}}{3} - \frac{x^8}{8} + c$$

Exercises

1. a) Find $\int 8x^5 + 3x^2\,\mathrm{d}x$, b) $\int \frac{2}{3}x\,\mathrm{d}x$.
2. Find $\int 3\cos x + 7x^3\,\mathrm{d}x$.
3. Find $\int 7x^{-2}\mathrm{d}x$.
4. Find $\int \frac{5}{x}\mathrm{d}x$.
5. Find $\int \frac{x+\cos 2x}{3}\mathrm{d}x$.
6. Find $\int 5\mathrm{e}^{4x}\mathrm{d}x$.
7. Find $\int \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\mathrm{d}x$.

Rule 3

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

for all real numbers n, except $n = -1$.

When $n = -1$, $\int x^n dx$ becomes $\int x^{-1} dx = \int \frac{1}{x} dx$. We don't need to worry that the rule above doesn't apply in this case, because we already know the integral of $\frac{1}{x}$.

Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \text{we have} \qquad \int \frac{1}{x} dx = \ln x + c.$$

Examples

Find

i $\int x^3 dx$

ii $\displaystyle\int \frac{dx}{x^2}$

iii $\int \sqrt{x} dx$

Solutions

i $\displaystyle\int x^3 dx = \frac{1}{(3+1)}x^4 + c = \frac{1}{4}x^4 + c.$

ii $\displaystyle\int \frac{dx}{x^2} = \int x^{-2} dx = \frac{1}{-2+1}x^{-2+1} + c = -\frac{1}{x} + c.$

iii $\displaystyle\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c.$

Exercises

1. Find anti-derivatives of the following functions:

i	x^5	**ii**	x^9
iii	x^{-4}	**iv**	$\frac{1}{x^2}$
v	$\frac{1}{\sqrt{x}}$	**vi**	$\sqrt[3]{x}$
vii	$x^{\sqrt{2}}$	**viii**	$x\sqrt{x}$
ix	$\frac{1}{x^\pi}$		

2. Find the following integrals:

i $\int -3x dx$

ii $\int \left(x^3 + 3x^2 + x + 4\right) dx$

iii $\int \left(x - \dfrac{1}{x}\right) dx$

iv $\int \left(\dfrac{2}{\sqrt{x}} + \dfrac{\sqrt{x}}{2}\right) dx$

PROPERTIES OF INTEGRAL : -

$$\int_a^b dx = b - a \Rightarrow \int_a^b c.dx = c(b-a)$$

$$\int_a^b c.f(x)dx = c.\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad ,c \in [a,b]$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$if\ f(x) \geq 0 \Rightarrow \int_a f(x)dx \geq 0$$

$$if\ f(x) \leq g(x) \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad ,n \neq -1$$

$$\int_{2}^{5}(x+5)dx = \frac{x^2}{2}+5x\bigg|_{2}^{5} = 25.5$$

$$\int_{0}^{1}(x^2+3x)dx = \frac{x^3}{3}+\frac{3}{2}x^2\bigg|_{0}^{1} = \frac{11}{6}$$

$$\int_{1}^{3}(2x^2+5x)^2(4x+5)dx = \frac{(2x^2+5x)^3}{3}\bigg|_{1}^{3} = ?$$

$$\int_{1}^{3}(2x^2+6x)^2(2x+3)dx = \frac{1}{2}\int_{1}^{3}(2x^2+6x)^2(4x+6)dx = \frac{(2x^2+6x)^3}{3}\bigg|_{1}^{3} = ?$$

$$\int_{0}^{1}(x^3+1)^2dx = \int_{0}^{1}(x^6+2x^3+1)dx = \frac{x^7}{7}+\frac{x^4}{2}+x\bigg|_{0}^{1} = ?$$

<u>EXAM</u> : -

$$\int_{6}^{10}dx = 10-6 = 4$$

$$\int_{6}^{10}5.dx = 5(10-6) = 5.4 = 20$$

$$\int_{1}^{3}x\,dx = \frac{x^2}{2}\bigg|_{1}^{3} = \frac{1}{2}(9-1) = 4$$

$$\int_{2}^{5}(x+5)dx = \frac{x^2}{2}+5x\bigg|_{2}^{5} = 25.5$$

$$\int_{0}^{1}(x^2+3x)dx = \frac{x^3}{3}+\frac{3}{2}x^2\bigg|_{0}^{1} = \frac{11}{6}$$

$$\int_{1}^{3}(2x^2+5x)^2(4x+5)dx = \frac{(2x^2+5x)^3}{3}\bigg|_{1}^{3} = ?$$

$$\int_{1}^{3}(2x^2+6x)^2(2x+3)dx = \frac{1}{2}\int_{1}^{3}(2x^2+6x)^2(4x+6)dx = \frac{(2x^2+6x)^3}{3}\bigg|_{1}^{3} = ?$$

$$\int_{0}^{1}(x^3+1)^2dx = \int_{0}^{1}(x^6+2x^3+1)dx = \frac{x^7}{7}+\frac{x^4}{2}+x\bigg|_{0}^{1} = ?$$