Remarke :

If $\underline{X} \sim N(\mu, \Sigma)$, then $Y = CX + d$ where (d) is a vector of constants of size $(P \times 1)$ follow $N_p(C\underline{\mu} + d, C\Sigma C')$

Proof

$$M_{\underline{Y}}(t) = E e^{\underline{t}'Y} = E e^{\underline{t}'(C\underline{X}+\underline{d})} = E e^{\underline{t}'C\underline{X} + \underline{t}'\underline{d}}$$

$$= E e^{\underline{t}'\underline{d}} e^{t'C\underline{X}} , = e^{t'd} E e^{t'C\underline{X}}$$

$$= e^{\underline{t}'\underline{d}} E e^{\underline{t}^{*}\underline{X}} = e^{\underline{t}'\underline{d}} M_{\underline{X}}(t^{*})$$

$$= e^{t'd} e^{t'C\underline{\mu} + \frac{1}{2} t'C\Sigma C' t}$$

$$= e^{\underline{t}'\underline{d} + t'C\underline{\mu} + \frac{1}{2} t'C\Sigma C' t}$$

$$= e^{t'\underbrace{(C\underline{\mu}+d)}_{\mu} + \frac{1}{2} t'\underbrace{C\Sigma C'}_{\Sigma} t}$$

which is the m.g.f of $Y \sim N_p(C\underline{\mu}+d, C\Sigma C')$

$$\therefore f(X) = \frac{1}{(2\pi)^{p/2} |C\Sigma C'|^{\frac{1}{2}}} e^{-\frac{1}{2}[Y-(C\mu+d)]'(C\Sigma C')^{-1} \times [Y-(C\mu+d)]}$$

Example

If $\underline{X} \sim N_2 \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{bmatrix} 2 & 1 \\ & 3 \end{bmatrix} \right\}$ find p.d.f of

$$Y = \begin{bmatrix} 2X_1 + X_2 + 3 \\ X_1 + X_2 - 2 \end{bmatrix}$$

$$Y = C\underline{X} + \underline{d}$$

$$Y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$E(Y) = C\underline{\mu} + \underline{d} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix} = \begin{bmatrix} EY_1 \\ EY_2 \end{bmatrix}$$

$$v\text{-}cov(\underline{Y}) = C\Sigma C' = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 10 & 7 \end{bmatrix}$$

Q.) what the second moment of Y_1 around zero

$$var(Y_1) = E(X_1)^2 - \{E(X_1)\}^2$$

$$15 = EX_1^2 - (10)^2 \Rightarrow EY_1^2 = 15 + 100 = \boxed{115}$$

Q2) what the first moment of Y_2 around zero.

$$E(Y_2) = 3$$

Q) find the EY_1Y_2

$$cov(Y_1, Y_2) = EY_1Y_2 - EY_1EY_2$$

$$10 = EY_1Y_2 - (10)(3) \Rightarrow EY_1Y_2 = 10 + 30 = 40$$

Some Properties of m.g.f.

① $M_{\underline{X}}(0,0,\dots,0) = 1$

② $M_{\underline{X}}(t_1,0,\dots,0) = M_{X_1}(t_1)$

③ $M_{\underline{X}}(t_1,t_2,\dots,t_q,0,\dots,0) = M_{\underline{X}}(t_1,t_2,\dots,t_q)$

$= M_{X_1,X_2,\dots,X_q}(t_1,t_2,\dots,t_q)$

④ $EX_1 = \left.\dfrac{\partial M_X(\underline{t})}{\partial t_1}\right|_{t_1=t_2=\dots=t_p=0}$

④ $EX_1^2 = \left.\dfrac{\partial^2 M_X(\underline{t})}{\partial t_1^2}\right|_{t_1=t_2=\dots=t_p=0}$

⑤ $\sigma_{X_1}^2 = \left.\dfrac{\partial^2 M_{\underline{X}}(\underline{t})}{\partial t_1^2}\right|_{t_1=t_2=\dots=t_p=0} - \left[\left.\dfrac{\partial M_X(\underline{t})}{\partial t_1}\right|_{t_1=t_2=\dots=t_p=0}\right]^2$

⑥ $EX_1X_2 = \left.\dfrac{\partial^2 M_{\underline{X}}(\underline{t})}{\partial t_1\,\partial t_2}\right|_{t_1=t_2=\dots=t_p=0}$

⑦ $\mathrm{Cov}(X_1,X_2) = \left.\dfrac{\partial^2 M_{\underline{X}}(\underline{t})}{\partial t_1\,\partial t_2}\right|_{t_1=t_2=\dots=t_p=0} - \left[\left.\dfrac{\partial M_{\underline{X}}(\underline{t})}{\partial t_1}\right|_{t_1=\dots=t_p=0} * \left.\dfrac{\partial M_{\underline{X}}(\underline{t})}{\partial t_2}\right|_{t_1=\dots=t_p=0}\right]$

⑧ If $X_1, X_2, \dots, X_p$ are indep. r.v.'s then

$M_{\underline{X}}(\underline{t}) = \prod_{i=1}^{p} M_{X_i}(t_i)$

⑨ If $X_1 = X_2 = \dots = X_p$ r.v.'s then

$M_{\underline{X}}(\underline{t}) = [M_{X_1}(t_1)]^p$ or f. ... v