Interplation and Extraplation

Interplation means to find approximate values of a function $y=f(x)$ for an x between different x-values $x_0, x_1, x_2, \dots, x_n$ at which the values of $f(x)$ are given $y_0=f(x_0), y_1=f(x_1), \dots, y_n=f(x_n)$.
A standard idea in interplation is to find a polynomial $P_n(x)$ of degree (n) or less than (n) that assumes the given values; thus

$$P_n(x_0)=y_0 \quad , P_n(x_1)=y_1, \dots, P_n(x_n)=y_n$$

where $P_n(x)$ an <u>interpolation polynomial</u> and $x_0, x_1, \dots, x_n$ the <u>nodes</u>.
We use $P_n(x)$ to get approximate values of $f(x)$ for x's between x_0 and x_n (<u>interpolation</u>) or sometimes outside the given interval (<u>extrapolation</u>).

1) Lagrange Interpolation Polynomial

Let $y=f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$. Since there are $(n+1)$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in (x) of degree (n) defined as a formula

$$P_n(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \dots + f(x_n)L_n(x)$$
$$= \sum_{i=0}^{n} f(x_i)\, L_i(x) \qquad \text{———} (1)$$

and

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

$$= \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \qquad (2)$$

Ex (1):- For the following tables of values:

x	1	2	3	4
$f(x)$	1	8	27	64

Find $f(3.5)$ using a Lagrange interpolation polynomial with $n=2$.

Sol:- Since $n=2$ we shall use only three values

$x_1 = 2$, $x_2 = 3$, $x_3 = 4$

$y_1 = 8$, $y_2 = 27$, $y_3 = 64$

$$P_2(x) = \sum_{i=1}^{n=2} f(x_i) L_i(x)$$

$$= f(x_1) L_1(x) + f(x_2) L_2(x) + f(x_3) L_3(x)$$

$$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-3)(x-4)}{(2-3)(2-4)} = \frac{1}{2}(x-3)(x-4)$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-2)(x-4)}{(3-2)(3-4)} = -(x-2)(x-4)$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-2)(x-3)}{(4-2)(4-3)} = \frac{1}{2}(x-2)(x-3)$$

$$P_2(x) = (8)\left(\frac{1}{2}(x-3)(x-4)\right) + (27)\left(-(x-2)(x-4)\right) + (64)\left(\frac{1}{2}(x-2)(x-3)\right)$$

$$= 4(x^2-7x+12) - 27(x^2-6x+8) + 32(x^2-5x+6)$$

$$= 4x^2-28x+48-27x^2+162x-216+32x^2-160x+192$$

$$= 9x^2-26x+24$$

$$P_2(3.5) = 9(3.5)^2-26(3.5)+24 = 43.25$$

Ex(2) :- Use the Lagrange iterpolation polynomial to estimate the values $f(2)$ and $f(5)$ from the following data.

x	0	1	3
$f(x)$	-1	2	7

sol :- Since the number of data are (3)

then $n = 2$

$$P_2(x) = \sum_{i=0}^{n=2} f(x_i) L_i(x)$$

$$= f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}(x)(x-3)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}(x)(x-1)$$

$$P_2(x) = (-1)\left(\frac{1}{3}(x-1)(x-3)\right) + (2)\left(-\frac{1}{2}(x)(x-3)\right) + (7)\left(\frac{1}{6}(x)(x-1)\right)$$

$$= -\frac{1}{3}(x^2-4x+3) - (x^2-3x) + \frac{7}{6}(x^2-x)$$

$$= -\frac{1}{6}x^2 + \frac{19}{6}x - 1$$

$$P_2(2) = -\frac{1}{6}(2)^2 + \frac{19}{6}(2) - 1 = -\frac{2}{3} + \frac{19}{3} - 1 = \frac{14}{3} \simeq f(2)$$

$$P_2(5) = 10.6667 \simeq f(5)$$

H.W. :- Use Lagrange interpolation polynomial to find the value of y when $x = 10$, if the following values of x and y are given:

x	5	6	9	11
y	12	13	14	16

$f(10) \simeq 14.67$

Algorithm of Lagrange Approximation

Input: n, x_i, x^*, $y_i = f(x_i)$ for all $i = 0, 1, \ldots, n$

step(1): Compute

$$L_i(x^*) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x^* - x_j)}{(x_i - x_j)}$$

step(2): Set

$$P(x^*) = \sum_{i=0}^{n} f(x_i) L_i(x^*)$$

step(3): Print the value $P(x^*)$.