The Conditional distribution

we have multivariate normal vector $\underline{X} \sim N(\mu, \Sigma)$. Consider partitioning μ and x into:

$$\underline{X} = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}$$

With similar partition of Σ into:

$$\Sigma = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right]$$

Σ_{11} : var - cov matrix of $x^{(1)}$

Σ_{22} : var - cov matrix of $x^{(2)}$

Σ_{12} : covariance between $x^{(1)}$ & $x^{(2)}$

There are two cases :-

Case (1) : $x^{(1)}$ and $x^{(2)}$ are independent

$\Sigma_{12}' = \Sigma_{21} = 0$

$f(\underline{x}^{(1)}, \underline{x}^{(2)}) = f(\underline{x}^{(1)}) \cdot f(\underline{x}^{(2)})$

②

$$= \frac{1}{(2\pi)^{q/2}\,|\Sigma_{11}|^{1/2}}\, e^{-\frac{1}{2}(x^{(1)}-\mu^{(1)})'\Sigma_{11}^{-1}(\underline{x}^{(1)}-\mu^{(1)})} \cdot \frac{1}{(2\pi)^{\frac{p-q}{2}}\,|\Sigma_{22}|^{1/2}}$$

$$\cdot\, e^{-\frac{1}{2}(x^{(2)}-\mu^{(2)})'\Sigma_{22}^{-1}(\underline{x}^{(2)}-\mu^{(2)})}$$

$$\therefore f(x^{(1)} \mid \underline{x}^{(2)} = \underline{x}^{(2)}) = \frac{f(\underline{x}^{(1)}, \underline{x}^{(2)})}{f(\underline{x}^{(2)} = \underline{x}^{(2)})} = f(x^{(1)})$$

$$= \frac{1}{(2\pi)^{q/2}\,|\Sigma_{11}|^{1/2}}\, e^{-\frac{1}{2}(x^{(1)}-\mu^{(1)})\Sigma_{11}^{-1}(x^{(1)}-\mu^{(1)})}$$

$$\therefore (\underline{X}^{(1)} \mid \underline{X}^{(2)} = \underline{x}^{(2)}) \sim N(\underline{\mu}^{(1)}, \Sigma_{11})$$

Case (2): $(X^{(1)})$ & $(X^{(2)})$ are Correlated. We have the following Theorem :-

Theorem:

مركبات - مكونات

Let the Components of $\underline{X}$ be divided into two groups Composing subvectors $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$.
suppose the mean μ is similarly divided into $\mu^{(1)}$ & $\mu^{(2)}$, and suppose the covariance matrix Σ of X is divided into Σ_{11}, Σ_{12} and Σ_{22}, if the distribution of $\underline{X}$ is multivariate normal dist. then the Conditional distribution

of $X^{(1)}$ given $X^{(2)} = x^{(2)}$ is multi-normal with mean $\mu^{(1)} + \Sigma_{12} \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)})$ and

Covariance matrix $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$

Proof:

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Therefore we make the following Linear trasformation which generates the independence.

$$Y^{(1)} = X^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} X^{(2)} \Rightarrow E(Y^{(1)}) = \mu^{(1)} - \Sigma_{12} \Sigma_{22}^{-1} \mu^{(2)} = \vartheta^{(1)}$$

$$Y^{(2)} = X^{(2)} \Rightarrow E(Y^{(2)}) = \mu^{(2)} = \vartheta^{(2)}$$

Since $Y^{(1)}$, $Y^{(2)}$ are independent then

$$f(Y^{(1)}, Y^{(2)}) = f(Y^{(1)})\, f(Y^{(2)})$$

$$\therefore f(Y^{(1)}, Y^{(2)}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{1/2}} e^{-\frac{1}{2}[Y^{(1)} - \vartheta^{(1)}]' \Sigma_{11.2}^{-1} [Y^{(1)} - \vartheta^{(1)}]}$$

$$* \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{1/2}} e^{-\frac{1}{2}[Y^{(2)} - \vartheta^{(2)}]' \Sigma_{22}^{-1} [Y^{(2)} - \vartheta^{(2)}]}$$

To find $f(X^{(1)} / X^{(2)} = x^{(2)})$, first we must find $f(X^{(1)}, X^{(2)})$

The joint dist.^{n} of $X^{(1)}, X^{(2)}$ can be found by the transformation technique as follows:

$$f(\underline{X}) = f(\underline{X}^{(1)}, \underline{X}^{(2)}) = f(\underline{Y}^{(1)}, \underline{Y}^{(2)}) \cdot |J|$$

$$|J| = \begin{vmatrix} \frac{\partial \underline{Y}^{(1)}}{\partial \underline{X}^{(1)}} & \frac{\partial \underline{Y}^{(1)}}{\partial X^{(2)}} \\ \frac{\partial Y^{(2)}}{\partial \underline{X}^{(1)}} & \frac{\partial Y^{(2)}}{\partial X^{(2)}} \end{vmatrix} = \begin{vmatrix} I_{q\times q} & \Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{(p-q)(p-q)} \end{vmatrix}$$

$$= |I_{q\times q}| \, |I_{(p-q)(p-q)}| = \underline{\underline{1}}$$

$$\therefore f(X^{(1)}, X^{(2)}) = f(Y^{(1)}, Y^{(2)})$$

Since $Y^{(1)}$ & $Y^{(2)}$ are independent (cistém)

$$\therefore f(X^{(1)}, X^{(2)}) = f(Y^{(1)}) \cdot f(Y^{(2)})$$

subsitute the $Y^{(1)}$ & $Y^{(2)}$ values into the above equation we get:

$$\therefore f(\underline{X}^{(1)}, X^{(2)}) = \left\{ \frac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{1/2}} \exp\left[-\frac{1}{2} (X^{(1)} - \Sigma_{12}\Sigma_{22}^{-1} X^{(2)} - \mathcal{V}^{(1)})' \Sigma_{11.2}^{-1} (X^{(1)} - \Sigma_{12}\Sigma_{22}^{-1} X^{(2)} - \mathcal{V}^{(1)}) \right] \right\} * \left\{ \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{1/2}} \exp\left[-\frac{1}{2} * (X^{(2)} - \mathcal{V}^{2})' \Sigma_{22}^{-1} (X^{(2)} - \mathcal{V}^{2}) \right] \right\}$$

whereas: $\nu^{(1)} = \mu^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\mu^{(2)}$

$$\nu^{(2)} = \mu^{(2)}$$

$$\therefore L(\underline{X}^{(1)}, \underline{X}^{(2)}) = \left\{ \frac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{1/2}} \exp\left\{ -\frac{1}{2}\left[(\underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}X^{(2)}) - (\mu^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\mu^{(2)}) \right]' \Sigma_{11.2}^{-1} \left[X^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}X^{(2)} - (\mu^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\mu^{(2)}) \right] \right\} \right\} \times \left\{ \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2}(X^{(2)} - \mu^{(2)})' \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)}) \right\} \right\}$$

$$= \left\{ \frac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{\frac{1}{2}}} e^{-\frac{1}{2}\left\{ X^{(1)} - [\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(X^{(2)} - \mu^{(2)})] \right\}' \Sigma_{11.2}^{-1} [X^{(1)} - [\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(X^{(2)} - \mu^{(2)})]} \right\} * \left\{ \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{\frac{1}{2}}} e^{-\frac{1}{2}[\underline{X}^{(2)} - \mu^{(2)})' \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)})]} \right\}$$

The conditional dist. of $(\underline{X}^{(1)} \mid \underline{X}^{(2)} = \underline{x}^{(2)})$ may be defined as :-

$$L(\underline{X}^{(1)} / \underline{X}^{(2)} = \underline{x}^{(2)}) = \frac{L(\underline{X}^{(1)}, X^{(2)})}{L(X^{(2)})}$$

$$\therefore L(X^{(1)} \mid X^{(2)} = x^{(2)}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{1/2}} e^{-\frac{1}{2}\left\{ \underline{x}^{(1)} - [\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(X^{(2)} - \mu^{(2)})]' \Sigma_{11.2}^{-1} [x^{(1)} - [\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})] \right\}}$$