Chapter Three
Numerical Solution of Ordinary Differential Equations

Theorem (Existence and uniqueness)

Assume that $f(x, y)$ is continuous function in a region $R=\{(x,y): a \le x \le b,\ c \le y \le d\}$. If f satisfies a Lipschitz condition on R in the variable y and $(x_0,y_0) \in R$, then the initial value problem $y' = f(x,y)$ with $y(x_0)=y_0$ has a unique solution $y=y(x)$ on some subinterval $x_0 \le x \le x_0+\delta$.

The Lipschitz condition is

$|f(x, y_1) - f(x, y_2)| \le K\,|y_1 - y_2|$, *all* $(x, y_1), (x, y_2)$ *in* R and

$|\partial f(x, y)/\partial y| \le k$

The general form of differential equation of order (n) is

$$y^{(n)} = f\left(x, y, y', y'', \dots, y^{(n-1)}\right)$$

In this chapter we solve the ordinary differential equation of order one (Initial value problem) which has the form:

$$y' = f(x,y)\ , \qquad y(x_0) = y_0 \qquad (1)$$

Let $[a, b]$ be the interval over which we want to find the solution to the well-posed I.V.P. $y' = f(x, y)$ with $y(x_0) = y_0$. In actuality, we will not find a differentiable function that satisfies the I.V.P. Instead, a set of points $\{(x_k, y_k)\}$ is generated, and the points are used for an approximation (i.e., $y(x_k) \approx y_k$). How can we proceed to:

First we subdivide the interval $[a, b]$ into n equal subintervals and select the mesh points $x_k = a + kh$ for $k = 0, 1, \ldots, n$ where $h = (b - a)/n$

The value h is called the step size. We now proceed to solve approximately equation (1) over $[x_0, x_n]$ with $y(x_0) = y_0$.

Numerical Methods:

1- **Euler's Method**

Assume that $y(x)$, $y'(x)$, and $y''(x)$ are continuous and use Taylor's theorem to expand $y(x)$ about $x = x_0$. For each value x there exists a value c_1 that lies between x_0 and x so that

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + y''(c_1)(x - x_0)^2/2! \qquad (2)$$

When $y'(x_0) = f(x_0, y(x_0))$ and $h = x_1 - x_0$ are substituted in eq.(2) the result is an expression for $y(x_1)$:

$$y(x_1) = y(x_0) + h\, f(x_0, y(x_0)) + y''(c_1)\frac{h^2}{2}$$

i.e.

$$y_1 = y_0 + h\, f(x_0, y_0)$$

and the local truncation error of Euler method is $O(h^2)$ (L.T.E.= $y''(c_1)\frac{h^2}{2}$)

The process is repeated and generates a sequence of points that approximates the solution curve $y = y(x)$. The general step for Euler's method is

$$x_{k+1} = x_k + h$$

$$y_{k+1} = y_k + h\, f(x_k, y_k) \quad \text{for} \quad k = 0, 1, \ldots, n-1 \qquad (3)$$

Example (1): Use Euler's method to solve approximately the initial value problem

$$\frac{dy(x)}{dx} = x - y, \quad y(0) = 1$$

At the values x=0.1 , 0.2 , 0.3 , 0.4

Solution:

$f(x,y)=x-y$, $x_0=0$, $y_0=1$, $h=0.1$

$x_{k+1} = x_k + h$

$y_{k+1} = y_k + h\, f(x_k , y_k)$ for $k = 0, 1, 2, 3$

k=0 $\longrightarrow$ $y(x_1)=y(0.1)=y_1= y_0 + h\, f(x_0 , y_0)$

$=1+(0.1)(x_0-y_0)=1+(0.1)(0-1)=0.9$

k=1 $\longrightarrow$ $y(x_2)=y(0.2)=y_2= y_1 + h\, f(x_1 , y_1)$

$=0.9+(0.1)(x_1-y_1)=0.9+(0.1)(0.1-0.9)=0.82$

k=2 $\longrightarrow$ $y(x_3)=y(0.3)=y_3= y_2 + h\, f(x_2 , y_2)$

$=0.82+(0.1)(x_2-y_2)=0.82+(0.1)(0.2-0.82)=0.758$

k=3 $\longrightarrow$ $y(x_4)=y(0.4)=y_4= y_3 + h\, f(x_3 , y_3)$

$=0.758+(0.1)(x_3-y_3)=0.758+(0.1)(0.3-0.758)=0.7122$

The exact solution is $y=2e^{-x}+x-1$

x_k	Euler's approximation $y(x_k)$	Exact solution	Error
0	1	1	0
0.1	0.9	0.909675	0.009675
0.2	0.82	0.837462	0.017462
0.3	0.758	0.781636	0.023636
0.4	0.7122	0.74064	0.02844

Example (2): Use Euler's method to approximate the solution for the initial value problem

$$y' = y - x^2 + 1 \quad , \quad y(0) = 0.5 \quad , \quad 0 \le x \le 1$$

with n=5.

Solution:

$f(x,y) = y - x^2 + 1$, $x_0=0$, $y_0=0.5$, n=5

$$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$$

$x_{k+1} = x_k + h = x_0 + kh = \frac{1}{5}k$

$x_1 = \frac{1}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{4}{5}$, $x_5 = 1$

$y_{k+1} = y_k + h\, f(x_k, y_k)$ for k = 0, 1, 2,3,4

k=0 $\longrightarrow$ $y(x_1) = y(\frac{1}{5}) = y_1 = y_0 + h\, f(x_0, y_0)$

$= 0.5 + (\frac{1}{5})(y_0 - x_0^2 + 1) = 0.5 + (\frac{1}{5})(0.5 - 0 + 1) = 0.8$

k=1 $\longrightarrow$ $y(x_2) = y(\frac{2}{5}) = y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.152$

k=2 $\longrightarrow$ $y(x_3) = y(\frac{3}{5}) = y_3 = y_2 + h\, f(x_2, y_2)$

$= 1.5504$

k=3 $\longrightarrow$ $y(x_4) = y(\frac{4}{5}) = y_4 = y_3 + h\, f(x_3, y_3)$

$= 1.9884$

k=4 $\longrightarrow$ $y(x_5) = y(1) = y_5 = y_4 + h\, f(x_4, y_4)$

$= 2.4587$