Remarke :-

If $\underline{X} \sim N_p(\mu, \Sigma)$ then $\underline{Y} = a\underline{X} \sim N(a\mu, a\Sigma a')$

where a is a $(1\times p)$ vector of constants.

Proof.

$$M_Y(\underline{t}) = E e^{\underline{t}'Y} = E e^{\underline{t}' a \underline{X}} = E e^{t^{*'} \underline{X}} = M_X(t^*)$$

$$= e^{t^{*'}\mu + \frac{1}{2} t^{*'} \Sigma t^*} = e^{t' a\mu + \frac{1}{2} t' a \Sigma a' t}$$

$\therefore Y \sim N(a\mu, a\Sigma a')$

ex Let $X \sim N_p(\mu, \Sigma)$ where $\underline{\mu} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\Sigma = \begin{bmatrix} 1 & 1 \\ & 2 \end{bmatrix}$

Find the p.d.f of $Y = 2X_1 + X_2$

$$Y = \underset{1\times 2}{(2 \quad 1)} \underset{2\times 1}{\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}}$$

$Y \sim N(a\mu, a\Sigma a')$

where $EY = \underline{a}'\underline{\mu} = (2 \quad 1)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 7$

$$var(Y) = a\Sigma a' = (2 \quad 1)\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= (2)^2(1) + (1)^2(2) - 2(2)(1) = 2$$

$\therefore Y \sim N(7, 2)$

$$f(Y) = \frac{1}{(2\pi)^{1/2}\sqrt{2}} e^{-\frac{1}{(2)(2)}\left(Y-7\right)^2}$$

and $f(x) = \frac{1}{(2\pi)(1)} e^{-\frac{1}{2}\begin{pmatrix} x_1-2 \\ x_2-3 \end{pmatrix}' \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1-2 \\ x_2-3 \end{pmatrix}}$

If we have

$Y_1 = 2X_1 + X_2$

$Y_2 = X_1 + X_2$

$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 2X_1 + X_2 \\ X_1 + X_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$\therefore C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \underline{Y} = C\underline{X}$

$\therefore \underline{Y} \sim N_2(C\underline{\mu}, C\Sigma C')$

$E(Y) = C\underline{\mu} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} EY_1 \\ EY_2 \end{bmatrix}$

$var(Y) = C\Sigma C' = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

If x_1, x_2 be two r.v.s.

$var(X_1 \mp X_2) = var(X_1) + var(X_2) \mp 2cov(X_1, X_2)$

$var(Y_1) = var(2X_1 + X_2) = 4var(X_1) + var X_2 + (2)(2)(1) cov(X_1, X_2)$

$var(Y_2) = var(X_1 + X_2) = var X_1 + var X_2 + 2cov(X_1, X_2)$

H.W. If $\underline{X} \sim N_2\left(\binom{2}{3}, \begin{bmatrix} 2 & 1 \\ & 3 \end{bmatrix}\right)$

find p.d.f. of

$$Y = \begin{bmatrix} 2X_1 + 3X_2 - 1 \\ X_1 + 2X_2 + 2 \end{bmatrix}$$

Remarke

If $\underline{X} \sim N(\mu, \Sigma)$, then $Y = cX + d$ where (d) is a vector of constants of size $(P \times 1)$ follow $N_p(c\underline{\mu} + d, c\Sigma c')$

<u>Proof</u>

$$M_{\underline{Y}}(t) = E e^{\underline{t}'Y} = E e^{\underline{t}'(c\underline{X} + d)} = E e^{\underline{t}'c\underline{X} + \underline{t}'\underline{d}}$$

$$= E e^{\underline{t}'\underline{d}} e^{t'c\underline{X}} = e^{t'd} E e^{t'c\underline{X}}$$

$$= e^{\underline{t}'\underline{d}} E e^{\underline{t}^{*}\underline{X}} = e^{t'\underline{d}} M_{\underline{X}}(t^{*})$$

$$= e^{t'd} e^{t'c\underline{\mu} + \frac{1}{2} t'c\Sigma c' t}$$

$$= e^{t'\underline{d} + t'c\mu + \frac{1}{2} t'c\Sigma c' t}$$

$$= e^{t'\underbrace{(c\mu + d)}_{\mu} + \frac{1}{2} t' \underbrace{c\Sigma c'}_{\Sigma} t}$$

which is the m.g.f of $Y \sim N_p(c\underline{\mu} + d, c\Sigma c')$

$$\therefore f(x) = \frac{1}{(2\pi)^{p/2} |c\Sigma c'|^{\frac{1}{2}}} e^{-\frac{1}{2}[Y - (c\mu + d)]'(c\Sigma c')^{-1} \times [Y - (c\mu + d)]}$$

Example

If $\underline{X} \sim N_2 \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{bmatrix} 2 & 1 \\ & 3 \end{bmatrix} \right\}$ find p.d.f of

$$Y = \begin{bmatrix} 2X_1 + X_2 + 3 \\ X_1 + X_2 - 2 \end{bmatrix}$$

$$Y = C\underline{X} + \underline{d}$$

$$Y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$E(Y) = C\underline{\mu} + \underline{d} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix} = \begin{bmatrix} EY_1 \\ EY_2 \end{bmatrix}$$

$$v\text{-}cov(\underline{Y}) = C \Sigma C' = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 10 & 7 \end{bmatrix}$$

Q.) what the second moment of Y_1 around zero

$$var(Y_1) = E(X_1)^2 - [E(X_1)]^2$$

$$15 = EX_1^2 - (10)^2 \longrightarrow EY_1^2 = 15 + 100 = \boxed{115}$$

Q) what the first moment of Y_2 around zero.

$$E(Y_2) = 3$$

Q) find the EY_1Y_2

$$cov(Y_1, Y_2) = EY_1Y_2 - EY_1EY_2$$

$$10 = EY_1Y_2 - (10)(3) \Rightarrow EY_1Y_2 = 10 + 30 = \boxed{40}$$