3-2 Cholesky Factorization:

If A is real, symmetric and positive definite matrix, then if has a unique factorization $A=L.L^T$ in which L lower triangular matrix with a positive diagonal.

A is symmetric if $A=A^T$ and positive definite if $X^TAX>0$ for every nonzero matrix X.

$$\ell_{11} = \sqrt{a_{11}}$$

$$\ell_{i1} = \frac{a_{i1}}{\ell_{11}} \qquad , for\ i = 2,3,\dots,n$$

$$\ell_{ii} = \left[a_{ii} - \sum_{k=1}^{i-1} \ell_{ik}^2\right]^{\frac{1}{2}} \quad , \qquad for\ \ i = 2,3,\dots,n-1$$

$$\ell_{ji} = \frac{1}{\ell_{ii}}\left[a_{ji} - \sum_{k=1}^{i-1} \ell_{jk}\ell_{ik}\right] \quad , \quad for\ j = i+1, i+2, \dots, n$$

$$\ell_{nn} = \left[a_{nn} - \sum_{k=1}^{n-1} \ell_{nk}^2\right]^{\frac{1}{2}}$$

Then we solve two systems

LY=B for Y and $L^TX=Y$ for X(solution) Where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.

Example 8: Factorization the following matrix A by using Cholesky Factorization.

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

Solution:

The matrix A is symmetric $(A=A^T)$ and positive definite then we can use Cholesky Factorization

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

$$\ell_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$\ell_{i1} = \frac{a_{i1}}{\ell_{11}} \qquad , for\ i = 2,3$$

$$\ell_{21} = \frac{a_{21}}{\ell_{11}} = \frac{-1}{2}$$

$$\ell_{31} = \frac{a_{31}}{\ell_{11}} = \frac{1}{2}$$

$$\ell_{ii} = \left[a_{ii} - \sum_{k=1}^{i-1} \ell_{ik}^2\right]^{\frac{1}{2}} \quad , \qquad for\ \ i = 2$$

$$\ell_{22} = [a_{22} - \ell_{21}^2]^{\frac{1}{2}} = \left[4.25 - (-0.5)^2\ \right]^{\frac{1}{2}} = 2$$

$$\ell_{ji} = \frac{1}{\ell_{ii}}\left[a_{ji} - \sum_{k=1}^{i-1} \ell_{jk}\ell_{ik}\right] \ , \quad for\ j = 3 \quad and \quad i\ = 2$$

$$\ell_{32} = \frac{1}{\ell_{22}}[a_{32} - \ell_{31}\ell_{21}] = \frac{1}{2}[2.75 - (0.5)(-0.5)] = 1.5$$

$$\ell_{nn} = \left[a_{nn} - \sum_{k=1}^{n-1} \ell_{nk}^2\right]^{\frac{1}{2}}, \quad n = 3$$

$$\ell_{33}\ = [a_{33} - (\ell_{31}^2 + \ell_{32}^2)]^{\frac{1}{2}} = \left[3.5 - ((0.5)^2\ + (1.5)^2\ )\right]^{\frac{1}{2}} = 1$$

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} . \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}$$