The marginal distribution   التوزيع الكلي

Let the r.v. $\underline{X} \sim N_p(\mu, \Sigma)$ is Partitioned into two groups of variants:-

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} \quad \text{where} \quad X^{(1)} = (X_1, X_2, \ldots, X_q)' \quad \text{and} \quad X^{(2)} = (X_{q+1}, X_{q+2}, \ldots, X_p)'$$

$$E(\underline{X}) = E\begin{pmatrix} \underline{X}^{(1)} \\ \hline \underline{X}^{(2)} \end{pmatrix} = \begin{pmatrix} \underline{\mu}^{(1)} \\ \hline \mu^{(2)} \end{pmatrix}$$

Where $\underline{\mu}^{(1)} = (\mu_1, \mu_2, \ldots, \mu_q)'$ and $\underline{\mu}^{(2)} = (\mu_{q+1}, \mu_{q+2}, \ldots, \mu_p)'$

also the v-cov. matrix is Partitioned as:-

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad \text{where}$$

$$\Sigma_{11} = Var(\underline{X}^{(1)}) = E(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(1)} - \underline{\mu}^{(1)})'$$

$$\Sigma_{22} = Var(\underline{X}^{(2)}) = E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(X^{(2)} - \mu^{(2)})'$$

$$\Sigma_{12} = Cov(\underline{X}^{(1)}, \underline{X}^{(2)}) = E(X^{(1)} - \mu^{(1)})(X^{(2)} - \mu^{(2)})'$$

$$\Sigma_{21} = Cov(\underline{X}^{(2)}, \underline{X}^{(1)}) = E(\underline{X}^{(2)} - \underline{\mu}^{(2)})(X^{(1)} - \underline{\mu}^{(1)})'$$

Note that $\Sigma_{12} \neq \Sigma_{21}$ but $\Sigma_{12} = \Sigma_{21}'$

we will find the marginal distribution in the following cases:

Case (1)

We will start with the simplest case in which $\underline{X}^{(1)}$ is uncorrelated with $\underline{X}^{(2)}$. This means Σ_{12} and Σ_{21} equal to zero matrix. Thus

$$\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}, \quad |\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22}|$$

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & \\ & \Sigma_{22}^{-1} \end{pmatrix}$$

The p.d.f of a r.v. $\underline{X}$ defined as:

$$f(\underline{X}) = f(X^{(1)}, X^{(2)}) = \frac{1}{(2\pi)^{p/2}|\Sigma|} e^{\left(-\frac{Q}{2}\right)} \quad \text{---(1)}$$

where: $Q = (\underline{X} - \underline{\mu})' \Sigma^{-1} (X - \mu)$

$$= \begin{pmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} \\ \underline{X}^{(2)} - \underline{\mu}^{(2)} \end{pmatrix}' \begin{pmatrix} \Sigma_{11}^{-1} & \\ & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} \underline{X}^{(1)} - \underline{\mu}^{(1)} \\ \underline{X}^{(2)} - \underline{\mu}^{(2)} \end{pmatrix}$$

$$\therefore Q = (X^{(1)} - \mu^{(1)})' \Sigma_{11}^{-1} (\underline{X}^{(1)} - \underline{\mu}^{(1)}) + (\underline{X}^{(2)} - \underline{\mu}^{(2)})' \Sigma_{22}^{-1} (\underline{X}^{(2)} - \underline{\mu}^{(2)}) \quad \text{---(2)}$$

substituting eq(2) in the eq(1) we get:-

$$f(\underline{X}^{(1)}, \underline{X}^{(2)}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(X^{(1)}-\mu^{(1)})' \Sigma_{11}^{-1} (X^{(1)}-\mu^{(1)})\right]$$

$$* \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\underline{X}^{(2)}-\underline{\mu}^{(2)})' \Sigma_{22}^{-1} (X^{(2)}-\mu^{(2)})\right\}$$

To find the marginal p.d.f of $(\underline{X}^{(1)})$ we integrate $f(\underline{X}^{(1)}, \underline{X}^{(2)})$ with respect to $(\underline{X}^{(2)})$

$$f(\underline{X}^{(1)}) = \int\int \cdots \int_{\Omega(\underline{X}^{(2)})} f(\underline{X}^{(1)}, \underline{X}^{(2)})\, d\underline{X}^{(2)}$$

$$f(X^{(1)}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11}|^{\frac{1}{2}}} \exp\left\{(X^{(1)}-\underline{\mu}^{(1)})' \Sigma_{11}^{-1} (\underline{X}^{(1)}-\underline{\mu}^{(1)})\right\}$$

$$* \int\int \cdots \int_{\Omega X^{(2)}} \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{\frac{1}{2}}} \exp\left\{X^{(2)}-\underline{\mu}^{(2)}) \Sigma_{22}^{-1} (\underline{X}^{(2)}-\underline{\mu}^{(2)})\right\}$$

$$\Rightarrow f(X^{(1)}) = \frac{1}{(2\pi)^{q/2} |\Sigma_{11}|^{\frac{1}{2}}} \exp\left\{(X^{(1)}-\mu^{(1)}) \Sigma_{11}^{-1} (\underline{X}^{(1)}-\underline{\mu}^{(1)})\right\}$$

$$\therefore \underline{X}^{(1)} \sim N_q(\mu^{(1)}, \Sigma_{11})$$

In the same manner (بنفس الطريقة) we can find the marginal p.d.f of $(\underline{X}^{(2)})$ where

$$f(X^{(2)}) = \int\int \cdots \int_{\Omega X^{(1)}} f(X^{(1)}, X^{(2)})\, d\underline{X}^{(1)}$$

$$\Rightarrow f(\underline{X}^{(2)}) = \frac{1}{(2\pi)^{\frac{p-q}{2}} \, |\Sigma_{22}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} (X^{(2)} - \mu^{(2)})' \, \Sigma_{22}^{-1} \, (X^{(2)} - \mu^{(2)}) \right\}$$

$$\Rightarrow \underline{X}^{(2)} \sim N_{p-q}(\underline{\mu}^{(2)}, \Sigma_{22})$$

$$\therefore f(\underline{X}^{(1)}, \underline{X}^{(2)}) \underset{\text{نستنتج}}{=} f(\underline{X}^{(1)}) \cdot f(\underline{X}^{(2)}) \quad \text{— } (3)$$

From eq (3) we conclude that $(\underline{X}^{(1)})$ and $(\underline{X}^{(2)})$ are independent and we can choose any variate to include in each group of variables. Thus we prove the sufficient condition of the following theorem:–

وبذلك نكون قد اثبتنا الشرط الكافي في النظرية الآتية

Theorem:

If $\underline{X} = (X_1, X_2, \ldots, X_p)' \sim N_p(\mu, \Sigma)$ then the necessity and sufficient conditions that may that the subgroup of variates to be independent of another group is that the covariance of any variable in first group and another variable in the seconde group equal to zero.

اذا كان المتجه X يتوزع توزيع طبيعي متوسطه μ ومصفوفة تباين Σ فان شرط الاستقلال لمجموعتين $\underline{X}^{(1)}$, $\underline{X}^{(2)}$ فانه الشرط الضروري والكافي لكي تكون المجموعة الاولى مستقلة عن المجموعة الثانية هو ان يكون تغاير اي عنصر في المجموعة الاولى مع اي عنصر في المجموعة الثانية مساوي صفر

To prove the necessity condition, let

$X_i \in (\underline{X}^{(1)})$ and $X_j \in X^{(2)}$, then

$$\sigma_{ij} = E(x_i - \mu_i)(x_j - \mu_j)$$

$$= \int_{x_i}\int_{x_j} (x_i - \mu_i)(x_j - \mu_j)\, f(x_i, x_j)\, dx_i\, x_j \quad \text{--- (1)}$$

but $f(x_i, x_j) = \underset{\text{except}(x_i, x_j)}{\int\int \cdots \int} f(x^{(1)}, x^{(2)})\, dx^{(1)}\, dx^{(2)}$

$$= \int_{x_1}\int_{x_2} \cdots \int_{x_{i-1}}\int_{x_{i+1}} \cdots \int_{x_q} f(x^{(1)})\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_q$$

$$\times \int_{x_{q+1}} \cdots \int_{x_{j-1}}\int_{x_{j+1}} \cdots \int_{x_p} f(x^{(2)})\, dx_{q+1}\, dx_{q+2} \cdots dx_{j-1}\, dx_{j+1} \cdots dx_p$$

$$\therefore f(x_i, x_j) = f(x^{(1)}) \cdot f(x^{(2)}) \quad \text{--- (2)}$$

Substitution eq (2) into eq (1) we get:

$$\sigma_{ij} = \int_{x_i} (x_i - \mu_i)\, f(x_i)\, dx_i \int_{x_j} (x_j - \mu_j)\, f(x_j)\, dx_j$$

$$= \left[\int_{x_i} x_i\, f(x_i)\, dx_i - \mu_i \underbrace{\int_{x_i} f(x_i)\, dx_i}_{1}\right]\left[\int_{x_j} x_j - \mu_j)\, f(x_j) - \mu_j \underbrace{\int_{x_j} f(x_j)\, dx_j}_{1}\right]$$

$$= [\mu_i - \mu_i][\mu_j - \mu_j] = 0$$

$$\therefore \sigma_{ij} = 0$$

$$\sigma_{ij} = \rho_{ij}\, \sigma_i\, \sigma_j \qquad \text{since } \sigma_i \neq 0 \text{ and } \sigma_j \neq 0$$

Therefor the result $\sigma_{ij} = 0$ equals to the result $\rho_{ij} = 0$. Thus the summary of the above theorem is that the independence of variate result from the uncorrelated variate. this result is tru in the multivariate normal distribution

لذلك فان قيمة $\sigma_{ij}=0$ تكافئ قيمة $\rho_{ij}=0$ وعليه فان خلاصة النظرية اعلاه تنص على ان الاستقلالية بين المتغيرات تكافئ عدم وجود ارتباط بين المتغيرات وهذه النتيجة تتحقق في حالة المتغيرات تتبع توزيع طبيعي متعدد

Ex: If $\underline{X} \sim N_4(\underline{\mu}, \Sigma)$ where $\underline{\mu} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$ and

$$\Sigma = \begin{bmatrix} 5 & 0 & -3 & 0 \\ 0 & 2 & 0 & -1 \\ -3 & 0 & 2 & 0 \\ 0 & -1 & 0 & 4 \end{bmatrix}$$

Find the marginal distribution of X_2, X_4

" " " " " X_1, X_3

Solu

$$\underline{X} = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix} \text{ where } \underline{X}^{(1)} = \begin{pmatrix} X_2 \\ X_4 \end{pmatrix} \text{ and } X^{(2)} = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$$

$$E(X^{(1)}) = \underline{\mu}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad E(X^{(1)}) = \underline{\mu}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Sigma = \begin{array}{c} \\ X_2 \\ X_4 \\ X_1 \\ X_3 \end{array} \begin{array}{c} \begin{array}{cccc} X_2 & X_4 & X_1 & X_3 \end{array} \\ \left[\begin{array}{cc|cc} 2 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ \hline 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{array}\right] \end{array} \qquad \begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array}$$

$$\operatorname{Var}(X^{(1)}) = \Sigma_{11} = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}, \quad |\Sigma_{11}| = 7, \quad |\Sigma_{11}|^{\frac{1}{2}} = \sqrt{7}$$

$$\operatorname{Var}(X^{(2)}) = \Sigma_{22} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}, \quad |\Sigma_{22}| = 1, \quad |\Sigma_{22}|^{\frac{1}{2}} = 1$$

$$\Sigma_{12} = 0, \quad \Sigma_{21} = 0$$

$$\Sigma_{11}^{-1} = \begin{pmatrix} \frac{4}{7} & \frac{1}{7} \\ & \frac{2}{7} \end{pmatrix}, \quad \Sigma_{22}^{-1} = \begin{pmatrix} 2 & 3 \\ & 5 \end{pmatrix}$$

$$\therefore f(X^{(1)}) = f(X_2, X_4) = \frac{1}{(2\pi)\sqrt{7}} \exp\left\{-\frac{1}{2}\left\{\frac{4}{7}(X_2-3)^2 + \frac{7}{2}(X_4-4)^2 + \frac{2}{7}(X_2-3)(X_4-4)\right\}\right.$$

$$f(X^{(4)}) = f(X_1, X_3) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[2(X_1-2)^2 + 5(X_2-1)^2 + 6X_1(X_1-1)\right]\right\}$$

$$\therefore X^{(4)} \sim N_2\left\{\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}\right.$$

$$X^{(2)} \sim N_2\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}\right\}$$

Case (2) :-

If the groups of variables $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ are correlated i.e. $\Sigma_{12} \neq 0$, $\Sigma_{21} \neq 0$. To find the marginal p.d.f. for each group we will transform the original variates $X_1, X_2, \ldots, X_p$ to new variates $Y_1, Y_2, \ldots, Y_p$ such that the groups $\underline{X}^{(1)}$ and $\underline{X}^{(2)}$ will be uncorrelated.
This can be done as follows :-

$$\underline{Y}^{(1)} = (Y_1, Y_2, \ldots, Y_q)'$$

$$\underline{Y}^{(2)} = (Y_{q+1}, Y_{q+2}, \ldots, Y_p)'$$

where

$$\underline{Y}^{(1)} = \underline{X}^{(1)} + \underline{M}\,\underline{X}^{(2)} \quad \text{--- (1)}$$

$$\underline{Y}^{(2)} = \underline{X}^{(2)} \quad \text{--- (2)}$$

where M is a matrix of degree $q \times (p-q)$ and it must be chosen such that $\underline{Y}^{(1)}$ and $\underline{Y}^{(2)}$ are uncorrelated i.e. :-

$$\text{Cov}(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = 0 \quad \text{--- (3)}$$

by taking the expected value to both sides of eq. (1) and eq. (2) we get:

$$E(\underline{Y}^{(1)}) = E(\underline{X}^{(1)} + \underline{M}\,X^{(2)}) = \underline{\mu}^{(1)} + M\underline{\mu}^{(2)}$$

$$E(\underline{Y}^{(2)}) = E(X^{(2)}) = \mu^{(2)}$$

$$\therefore \text{Cov}(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = E\left\{[(\underline{Y}^{(1)} - E(\underline{Y}^{(1)})][\underline{Y}^{(2)} - E(\underline{Y}^{(2)})]'\right\}$$

نعوض عن Y

$$= E\,(\underline{X}^{(1)} + M\underline{X}^{(2)} - \underline{\mu}^{(1)} - M\,\mu^{(2)})\,(\underline{X}^{(2)} - \underline{\mu}^{(2)})$$

نعوض عن Y

$$= E\,\{(\underline{X}^{(1)} - \underline{\mu}^{(1)}) + M(\underline{X}^{(2)} - \underline{\mu}^{(2)})\}\{\underline{X}^{(2)} - \underline{\mu}^{(2)}\}'$$

$$= E\,\{(\underline{X}^{(1)} - \underline{\mu}^{(1)})(\underline{X}^{(2)} - \underline{\mu}^{(2)})' + M(X^{(2)} - \mu^{(2)})(X^{(2)} - \mu^{(2)})'\}$$

$$\therefore Cov(\underline{Y}^{(1)}, \underline{Y}^{(2)}) = E(X^{(1)}-\mu^{(1)})(X^{(2)}-\mu^{(2)})' + M\,E(X^{(2)}-\mu^{(2)})(X^{(2)}-\mu^{(2)})'$$

$$Cov(Y^{(1)}, Y^{(2)}) = \Sigma_{12} + M\,\Sigma_{22}$$

بمساواة cov بالصفر نجد أن :

$$0 = \Sigma_{12} + M\,\Sigma_{22} \Rightarrow \Sigma_{12} = -M\,\Sigma_{22}$$

$$\therefore M = -\Sigma_{12}\Sigma_{22}^{-1} \quad \text{——— (5)}$$

Thus using eq(5) into eq(1) we get

$$\underline{Y}^{(1)} = X^{(1)} - \Sigma_{12}\Sigma_{22}^{-1} X^{(2)} \quad \text{——— (6)}$$

$$\underline{Y}^{(2)} = \underline{X}^{(2)} \quad \text{——— (7)}$$

The r.v. $\underline{Y} = (Y^{(1)}, Y^{(2)})$ follow a multivariate normal dist^{n}, This dist^{n} can be derived as follow eq (6) and eq (7) can be simplified as follows:-

$$Y = \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix} = \begin{pmatrix} I_q & -\Sigma_{12}\Sigma_{22}^{-1} \\ \underline{0} & I_{p-q} \end{pmatrix} \begin{pmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{pmatrix} \quad \text{——— 8}$$

$$E(\underline{Y}) = \begin{pmatrix} \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)} \\ \mu^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{pmatrix} = \begin{bmatrix} EY^{(1)} \\ EY^{(2)} \end{bmatrix}$$

$$\text{var cov}(Y) = E\begin{pmatrix} Y^{(1)} - \lambda^{(1)} \\ Y^{(2)} - \lambda^{(2)} \end{pmatrix} \left((\underline{Y}^{(1)} - \lambda^{(1)})' \quad (\underline{Y}^{(2)} - \underline{\lambda}^{(2)})' \right)$$

$V(Y^{(1)}) \longrightarrow V(X^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}X^{(2)})$

$$\text{var-cov}(Y) = \begin{bmatrix} E(Y^{(1)}-\lambda^{(1)})(Y^{(1)}-\lambda^{(1)})' & E(Y^{(1)}-\lambda^{(1)})(Y^{(2)}-\lambda^{(2)})' \\ E(Y^{(2)}-\lambda^{(2)})(Y^{(1)}-\lambda^{(1)}) & E(Y^{(2)}-\lambda^{(2)})(Y^{(2)}-\lambda^{(2)})' \end{bmatrix}$$

$V(X^{(1)})$ $\quad V(Y^{(1)})$ --- (4)

$$= \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} \Sigma_{11.2} & \\ & \Sigma_{22} \end{pmatrix}$$

where $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

From var-cov $(\underline{Y})$, we see that $Y^{(1)}$ and $Y^{(2)}$ are uncorrelated

the p.d.f of $\underline{Y}$ is defined as

$f(Y) = f(\underline{X})\,|J|$

From (1) and (2), we have

$\underline{X}^{(1)} = \underline{Y}^{(1)} - MY^{(2)}$ and $X^{(2)} = Y^{(2)}$

$$|J| = \begin{vmatrix} \frac{\partial X^{(1)}}{\partial Y^{(1)}} & \frac{\partial X^{(1)}}{\partial Y^{(2)}} \\ \frac{\partial X^{(2)}}{\partial Y^{(1)}} & \frac{\partial X^{(2)}}{\partial Y^{(2)}} \end{vmatrix} = \begin{vmatrix} I_q & \overbrace{-\Sigma_{12}\Sigma_{22}^{-1}}^{M} \\ 0 & I_{p-q} \end{vmatrix}$$

$$= |I_q|\,|I_{p-q}| = |I| = 1$$

$$\therefore f(\underline{Y}) = f(\underline{X}) \Big|_{\substack{X^{(1)} = Y^{(1)} - \Sigma_{12}\Sigma_{22}^{-1} Y^{(2)} \\ X^{(2)} = Y^{(2)}}}$$

$$= \frac{1}{(2\pi)^{P/2} \, |\Sigma_Y|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} (\underline{Y} - \underline{\lambda})' \Sigma_{Y}^{-1} (\underline{Y} - \underline{\lambda}) \right\}$$

$$= \frac{1}{(2\pi)^{P/2} \, |\Sigma_{11.2}|^{\frac{1}{2}} \, |\Sigma_{22}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \begin{pmatrix} \underline{Y}^{(1)} - \underline{\lambda}^{(1)} \\ \underline{Y}^{(2)} - \underline{\lambda}^{(2)} \end{pmatrix}' \right.$$

$$\left. \begin{pmatrix} \Sigma_{11.2}^{-1} & \\ & \Sigma_{22}^{-1} \end{pmatrix} \begin{pmatrix} \underline{Y}^{(1)} - \underline{\lambda}^{(1)} \\ \underline{Y}^{(2)} - \underline{\lambda}^{(2)} \end{pmatrix} \right\}$$

$$f(Y^{(1)}, Y^{(2)}) = \frac{1}{(2\pi)^{q/2} \, |\Sigma_{11.2}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} (\underline{Y}^{(1)} - \underline{\lambda}^{(1)})' \Sigma_{11.2}^{-1} (\underline{Y}^{(1)} - \underline{\lambda}^{(1)}) \right\}$$

$$* \frac{1}{(2\pi)^{\frac{P-q}{2}} \, |\Sigma_{22}|^{\frac{1}{2}}} \exp\left\{ \frac{-1}{2} (\underline{X}^{(2)} - \underline{\mu}^{(2)})' \Sigma_{22}^{-1} (X^{(2)} - \mu^{(2)}) \right\}$$

$$f(Y^{(1)}) = \int\int_{Y^{(2)}} \cdots \int f(\underline{Y}^{(1)}, \underline{Y}^{(2)}) \, dY^{(2)}$$

$$= \frac{1}{(2\pi)^{q/2} \, |\Sigma_{11.2}|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} (Y^{(1)} - \lambda^{(1)})' \Sigma_{11.2}^{-1} \right.$$

$$\left. (Y^{(1)} - \lambda^{(1)}) \right\}$$

$$\Rightarrow \underline{Y}^{(1)} \sim N_q\left(\underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)}, \Sigma_{11.2}\right)$$

In the same manner we conclude that

$$\underline{Y}^{(2)} \sim N_{p-q}\left(\underline{\mu}^{(2)}, \Sigma_{22}\right)$$

Thus we have proved the following Theorem

Theorem

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ then the marginal distribution for any group of some variates of a r.v. follow a multivariate normal with vector mean and variance-covariance matrix which they are partial from $\underline{\mu}$ and Σ

Ex Let $X \sim N_4(\underline{\mu}, \Sigma)$ where

$$\underline{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 4 & 1 & -1 & 2 \\ & 3 & 2 & 1 \\ & & 4 & 1 \\ & & & 2 \end{pmatrix}$$

and if $\underline{X}$ is partitioned into two groups

$X^{(1)} = (X_1 \quad X_2)'$ and $X^{(2)} = (X_3 \quad X_4)'$

Find the Probability distribution of the following

1. The Linear Combination $\underline{Y}^{(1)} = X^{(1)} + M X^{(2)}$ where $M = -\Sigma_{12}\Sigma_{22}^{-1}$
2. the marginal distn. of $X^{(2)}$
3. the Probability distn of $\underline{Y} = (\underline{Y}^{(1)}, \underline{Y}^{(2)})'$

Solu

$\underline{\mu}^{(1)} = (1 \quad 2)'$, $\underline{\mu}^{(2)} = (3 \quad 1)'$

$\Sigma_{11} = \begin{pmatrix} 4 & 1 \\ & 3 \end{pmatrix}$, $\Sigma_{22} = \begin{pmatrix} 4 & 1 \\ & 2 \end{pmatrix}$, $\Sigma_{12} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$

since $\Sigma_{12} \neq 0$ thus $X^{(1)}$ and $X^{(2)}$ are dependent group, and to find the marginal distn. of any variates of a r.v. $\underline{X}$, we must transform the r.v.

$\underline{X}$ to new independent r.v's.

① $\underline{Y}^{(1)} = \underline{X}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{X}^{(2)} \sim N_2(\underline{\lambda}^{(1)}, \Sigma_{11.2})$

where $\underline{\lambda}^{(1)} = \underline{\mu}^{(1)} - \Sigma_{12}\Sigma_{22}^{-1}\underline{\mu}^{(2)}$

$$= \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}-1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}\frac{2}{7} & \frac{-1}{7}\\ & \frac{4}{7}\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}\frac{10}{7}\\ \frac{3}{7}\end{pmatrix}$$

$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

$$= \begin{pmatrix}4 & 1\\ & 3\end{pmatrix} - \begin{pmatrix}\frac{-4}{7} & \frac{9}{7}\\ \frac{3}{7} & \frac{2}{7}\end{pmatrix}\begin{pmatrix}-1 & 2\\ 2 & 1\end{pmatrix} = \begin{pmatrix}\frac{6}{7} & \frac{6}{7}\\ & \frac{13}{7}\end{pmatrix}$$

$\therefore\ Y^{(1)} \sim N_2\left\{\begin{pmatrix}\frac{10}{7}\\ \frac{3}{7}\end{pmatrix}, \begin{pmatrix}\frac{6}{7} & \frac{6}{7}\\ & \frac{13}{7}\end{pmatrix}\right.$

② $\underline{Y}^{(2)} = \underline{X}^{(2)} \sim N_2(\mu^{(2)}, \Sigma_{22})$

$\rightarrow \underline{X}^{(2)} \sim N_2\left\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}4 & 1\\ & 2\end{pmatrix}\right\}$

③ $\underline{Y} \sim N_4(E(X), \Sigma_{\underline{Y}})$

where $E(\underline{Y}) = \left(E(\underline{Y}^{(1)})' \ E(\underline{Y}^{(2)})'\right)$

$$= \left(\frac{10}{7} \quad \frac{3}{7} \quad 3 \quad 1\right)'$$

and $\Sigma_{\underline{Y}} = \begin{pmatrix}\Sigma_{11.2} & 0\\ 0 & \Sigma_{22}\end{pmatrix} = \begin{pmatrix}\frac{6}{7} & \frac{6}{7} & & \\ \frac{6}{7} & \frac{13}{7} & & \\ & & 4 & 1\\ & & 1 & 2\end{pmatrix}$