(ii) Indirect method (or Iterative methods)

Definition 1:

A vector on R^n, the collection of all n-dimensional column vectors with real components, is a function, $\|.\|$ from R^n into R with the following properties:

i) $\|x\| \geq 0$ for all $x \in R^n$.

ii) $\|x\|=0$ if and only if $x=0$.

iii) $\|\alpha x\|=|\alpha|\,\|x\|$ for all $x \in R^n$ and $\alpha \in R$.

iv) $\|x+y\| \leq \|x\|+\|y\|$ for all $x,y \in R^n$.

Note:

If only i, iii, iv are satisfied, $\|.\|$ is called semi-norm.

Definition 2:

The L_1, L_2 and L_∞ norms for the vector $(x_1, x_2, \ldots, x_n)^t$ are defined by

L_1: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (First norm)

L_2: $\|x\|_2 = \sqrt{\sum_{n=1}^{n} x_i^2}$ (Euclidean norm)

and

L_∞: $\|x\|_\infty = \max_{1 \leq i \leq n} \{|x_i|\}$ (Infinite norm).

Example 9:

The vector $x=(-1,1,3)^t$ in has R^3 norms

$$\|x\|_1 = \sum_{i=1}^{3} |x_i| = |-1|+|1|+|3| = 5$$

$$\|x\|_2 = \sqrt{\sum_{n=1}^{3} x_i^2} = \sqrt{1+1+9} = \sqrt{11}$$

$$\|x\|_\infty = \max_{1 \leq i \leq 3} \{|x_i|\} = \max\{|-1|,|1|,|3|\} = 3$$

Definition 3:

If $x,y \in R^n$, the L_1, L_2 and L_∞ distance between x and y defined by

L_1: $\|x - y\|_1 = \sum_{i=1}^{n} |x_i - y_i|$ (First norm)

L_2: $\|x - y\|_2 = \sqrt{\sum_{n=1}^{n} (x_i - y_i)^2}$ (Euclidean norm)

and

L_∞: $\|x - y\|_\infty = \max_{1 \leq i \leq n} \{|x_i - y_i|\}$ (Infinite norm).

Iterative methods:

(1) Jacobi Method:

From the linear system of equations:

$$\begin{aligned}
&a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1\\
&a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2\\
&\vdots\\
&a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n=b_n
\end{aligned}$$

We generate the iterative equations:

$$x_1^{(k+1)}=\frac{b_1-a_{12}x_2^{(k)}-a_{13}x_3^{(k)}-\cdots-a_{1n}x_n^{(k)}}{a_{11}}$$

$$x_2^{(k+1)}=\frac{b_2-a_{21}x_1^{(k)}-a_{23}x_3^{(k)}-\cdots-a_{2n}x_n^{(k)}}{a_{22}}$$

$$\vdots$$

$$x_n^{(k+1)}=\frac{b_n-a_{n1}x_1^{(k)}-a_{n2}x_2^{(k)}-\cdots-a_{n,n-1}x_{n-1}^{(k)}}{a_{nn}}$$

Where k is number of iterations i.e. k=0, 1,…. We stop iteration if $\left|x_i^{(k+1)}-x_i^{(k)}\right|<\varepsilon$ for any k and all i=1, 2, …,n. To solve above system, we need an initial approximate solution $x^{(0)}=\begin{bmatrix}x_1^{(0)}\\x_2^{(0)}\\\vdots\\x_n^{(0)}\end{bmatrix}$ to the solution $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ and generates a sequence of vectors $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ that converges to x.

In general $x_i^{(k+1)}=\dfrac{b_i-\sum\limits_{\substack{j=1\\j\neq i}}^{n}a_{ij}x_j^{(k)}}{a_{ii}}$, i=1, 2, …, n; k=0, 1, … .

Note: The sufficient condition for convergence is $|a_{ii}|>\sum\limits_{\substack{j=1\\j\neq i}}^{n}|a_{ij}|$, i=1, 2, …, n.

Example 10:

Solve the system $4x-y+z=7$

$$4x-8y+z=-21$$
$$-2x+y+5z=15$$

by using Jacobi method with $x^{(0)}=y^{(0)}=z^{(0)}=0$.

Solution: Since $|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{3} |a_{ij}|$, i=1, 2, 3.

$$x^{(k+1)}=[7+y^{(k)}-z^{(k)}]/4$$
$$y^{(k+1)}=[-21-4x^{(k)}-z^{(k)}]/-8;\ k=0, 1, \ldots$$
$$z^{(k+1)}=[15+2x^{(k)}-y^{(k)}]/5$$

since $x^{(0)}=y^{(0)}=z^{(0)}=0$,

$\therefore$ $x^{(1)}=[7+y^{(0)}-z^{(0)}]/4=\frac{7}{4}$

$$y^{(1)}=[-21-4x^{(0)}-z^{(0)}]/-8=\frac{21}{8}$$
$$z^{(1)}=[15+2x^{(0)}-y^{(0)}]/5=3$$

If $\left|x^{(1)}-x^{(0)}\right|>\varepsilon$, $\left|y^{(1)}-y^{(0)}\right|>\varepsilon$ and $\left|z^{(1)}-z^{(0)}\right|>\varepsilon$ we must find $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$.

$$x^{(2)}=[7+y^{(1)}-z^{(1)}]/4=\left[7+\frac{21}{8}-3\right]/4=\frac{53}{32}$$
$$y^{(2)}=[-21-4x^{(1)}-z^{(1)}]/-8=[-21-7-3]/-8=\frac{31}{8}$$
$$z^{(1)}=[15+2x^{(0)}-y^{(0)}]/5=\left[15+\frac{7}{2}-\frac{21}{8}\right]/5=\frac{16}{5}$$

$\vdots$

And so on

Example 11:

Solve the system by using Jacobi method with accurate $\in = 10^{-6}$

$$4x_1 + 2x_2 + x_3 = 11$$
$$2x_1 + x_2 + 4x_3 = 16$$
$$-x_1 + 2x_2 \qquad = 3$$

With $x^{(0)}=(1,1,1)^t$

Algorithm of Jacobi Method

Input: n , a_{ij} , b_i ,x_{oi} , ε , N
Step (1): Set k=1
Step (2): If $k \leq N$compute the steps 3-6
Step (3): For i=1,2,…,n set

$$x_i = [b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_{oj}] / a_{ii}$$

Step (4): If $\|x - x_0\| \leq \varepsilon$ then x is the solution and stop
Step (5): Else set k=k+1
Step (6): Set x_{0i}=x_i , for i=1,2,…, n