

3- Taylor Series Method

The problem to be solved is a first order ODE

$$\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0$$

We use Taylor series expansions to the exact solution

$$y(x) = y(x_0) + (x - x_0) \left. \frac{dy}{dx} \right|_{\substack{x=x_0, \\ y=y_0}} + \frac{(x - x_0)^2}{2!} \left. \frac{d^2 y}{dx^2} \right|_{\substack{x=x_0, \\ y=y_0}} + \frac{(x - x_0)^3}{3!} \left. \frac{d^3 y}{dx^3} \right|_{\substack{x=x_0, \\ y=y_0}} + \dots$$

where

$$y' = f(x, y)$$

$$y'' = f'(x, y) = f_x + f_y y' = f_x + f \cdot f_y$$

$$y''' = f''(x, y) = f_{xx} + 2f \cdot f_{xy} + f \cdot (f_y)^2 + f^2 \cdot f_{yy} + f_x \cdot f_y$$

نلاحظ ان صيغة تيلر تحتاج الى حساب مشتقات الدالة $f(x, y)$ ومن رتب عالية وذلك للحصول على دقة جيدة . ان الصيغة العامة لمتسلسلة تيلر من الرتبة (k) (kth order Taylor series formula) هي:

$$y_{i+1} = y_i + h y'_i + \frac{h^2}{2!} y''_i + \dots + \frac{h^k}{k!} y_i^{(k)} + O(h^{k+1}) \quad (5)$$

Where $h = x - x_0$, $x_{i+1} = x_i + h$ And the local truncation error is

$$E_T = O(h^{k+1}) = \frac{h^{k+1}}{(k+1)!} y_i^{(k+1)}(\theta), \quad \theta \in (x_i, x_{i+1})$$

The Taylor series method of order k^{th} has the property that local truncation error (L.T.E.) is of order $O(h^{k+1})$, k can be chosen as large as necessary to make this error as small as desired

Example (5): Use Taylor series method of order(4) to solve approximately the initial value problem

$$y' = x - y, \quad y(0) = 1$$

At the values $x=0.1, 0.2, 0.3, 0.4$

Solution:

$$f(x,y)=x-y, \quad x_0=0, \quad y_0=1, \quad h=0.1$$

$$x_{i+1} = x_i + h$$

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!} y''_i + \frac{h^3}{3!} y^{(3)}_i + \frac{h^4}{4!} y^{(4)}_i, \quad \text{for } i=0, 1, 2, 3$$

$$i=0 \longrightarrow y_1 = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y^{(3)}_0 + \frac{h^4}{4!} y^{(4)}_0$$

$$y'_0 = x_0 - y_0 = -1$$

$$y''_0 = f'(x_0, y_0) = f_x + f_y y' = 1 - (x_0 - y_0) = 1 - x_0 + y_0 = 2$$

$$y'''_0 = -1 + x_0 - y_0 = -2$$

$$y^{(4)}_0 = 1 - x_0 + y_0 = 2$$

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y^{(3)}_0 + \frac{h^4}{4!} y^{(4)}_0$$

$$= 1 + 0.1 * -1 + \frac{(0.1)^2}{2!} * 2 + \frac{(0.1)^3}{3!} * (-2) + \frac{(0.1)^4}{4!} * 2 = 0.909675$$

$$i=1 \longrightarrow y_2 = y_1 + hy'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y^{(3)}_1 + \frac{h^4}{4!} y^{(4)}_1$$

$$y'_1 = 0.809675$$

$$y''_1 = 1.804675$$

$$y'''_1 = -1.809675$$

$$y^{(4)}_1 = 1.809675$$

$$y_2 = y_1 + hy'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y^{(3)}_1 + \frac{h^4}{4!} y^{(4)}_1 = 0.837462$$

Also $y_3 = 0.781636$, and $y_4 = 0.740640$

Example (6): Use Taylor series method of order(3) to obtain the numerical solution $y(2.1)$ of the initial value problem

$$y' = \frac{x-y}{x}, \quad y(2) = 2$$

Solution:

$$f(x,y) = \frac{x-y}{x} = 1 - \frac{y}{x}, \quad x_0=2, \quad y_0=2, \quad h=0.1$$

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!} y''_i + \frac{h^3}{3!} y^{(3)}_i$$

$$y(2.1) = y_1 = y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y^{(3)}_0$$

$$y'_0 = 1 - \frac{y_0}{x_0}$$

$$y''_0 = \frac{-1}{x_0} + \frac{2y_0}{x_0^2}$$

$$y'''_0 = \frac{3}{x_0^2} - \frac{6y_0}{x_0^3}$$

$$y(2.1) = 2.00238$$

Taylor Series Method Algorithm of order(k)

Input: $a, b, y_0, n, f(x,y)$

Step(1): compute $h = \frac{b-a}{n}$, set $x_0=a$

Step(2): For $i=0,1,2,\dots,n$

Step(3): Set $x_{i+1} = x_i + h$

Step(4): Find all the derivatives at the point (x_i, y_i) i.e. : $y', y'', y''', \dots y^{(k)}$

Step(5): Evaluate $y_{i+1} = y_i + hy'_i + \frac{h^2}{2!} y''_i + \dots + \frac{h^k}{k!} y^{(k)}_i$

Step(6): Print x_{i+1} and y_{i+1} for all i , then stop.