(2) Gauss-Seidel Method:

Rewrite the linear system Ax=b as

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}}{a_{22}}$$

$$\vdots$$

$$x_n^{(k+1)} = \frac{b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{n,n-1}x_{n-1}^{(k+1)}}{a_{nn}}$$

Where k is number of iterations i.e. k=0, 1,…. We stop iteration if $\left|x_i^{(k+1)} - x_i^{(k)}\right| < \varepsilon$ for any k and all i=1, 2, …,n. To solve above system, we need an initial approximate solution $x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}$ to the solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and generates a sequence of vectors $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ that converges to x.

In general $x_i^{(k+1)} = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}}{a_{ii}} = \dfrac{b_i - S_1 - S_2}{a_{ii}}$, i=1, 2, …, n; k=0, 1, … ;

where S_1=0 for i=1 and S_2=0 for i=n..

The sufficient condition for convergence is $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$, i=1, 2, …, n.

Example 12:

Solve the given system in Example 10 by Gauss-Seidel method.

Solution: Since $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{3} |a_{ij}|$, i=1, 2, 3.

$x^{(k+1)}=[7+y^{(k)}-z^{(k)}]/4$
$y^{(k+1)}=[-21-4x^{(k+1)}-z^{(k)}]/-8$; k=0, 1, 2, …
$z^{(k+1)}=[15+2x^{(k+1)}-y^{(k+1)}]/5$

let $x^{(0)}=y^{(0)}=z^{(0)}=0$,

$\therefore \quad x^{(1)}=[7+y^{(0)}-z^{(0)}]/4=\frac{7}{4}$

$y^{(1)}=[-21-4x^{(1)}-z^{(0)}]/-8=?$

$z^{(1)}=[15+2x^{(1)}-y^{(1)}]/5=?$

If $\left|x^{(1)}-x^{(0)}\right|>\varepsilon$, $\left|y^{(1)}-y^{(0)}\right|>\varepsilon$ and $\left|z^{(1)}-z^{(0)}\right|>\varepsilon$ we must find $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$.

$x^{(2)}=[7+y^{(1)}-z^{(1)}]/4=?$

$y^{(2)}=[-21-4x^{(2)}-z^{(1)}]/-8=?$

$z^{(1)}=[15+2x^{(2)}-y^{(2)}]/5=?$

$\vdots$

Algorithm of Gauss-Seidel Method

Input: n , a_{ij} , b_i ,x_{oi} , ε , N

Step (1): Set k=1

Step (2): If $k \leq N$compute the steps 3-6

Step (3): For i=1,2,…,n set

$$x_i = [b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_{oj}] / a_{ii}$$

Step (4): If $\|x - x_0\| \leq \varepsilon$ then x is the solution and stop

Step (5): Else set $k=k+1$

Step (6): Set $x_{0i}=x_i$, for i=1,2,…, n