

4- Runge-Kutta Methods

وضعت هذه الطريقة من قبل العالمين الالمانيين رانك وكوتا لتجنب حساب المشتقات العالية التي تتطلبها طريقة تيلر. وتعد طريقة رانك-كوتا من الرتبة الرابعة اكثرها شيوعا لسهولة استخدامها ورتبة الخطأ فيها عالية. تعرف صيغة رانك-كوتا من الرتبة الثانية (RK-Second Order Runge Kutta) ((2)) ذات خطأ البتر المحلي من الرتبة $O(h^3)$ بـ

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2} [f(x_i, y_i) + f(x_i+h, y_i + h f(x_i, y_i))] \\ &= y_i + \frac{1}{2} [k_1 + k_2] \end{aligned} \quad (6)$$

$$\text{where } k_1 = hf(x_i, y_i) \quad \text{and} \quad k_2 = hf(x_i + h, y_i + k_1)$$

وتعرف صيغة رانك-كوتا من الرتبة الرابعة (Fourth Order Runge-Kutta (RK-4)) بـ

$$y_{i+1} = y_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (7)$$

$$\begin{aligned} \text{where } k_1 &= hf(x_i, y_i) \\ k_2 &= hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1) \\ k_3 &= hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2) \\ k_4 &= hf(x_i + h, y_i + k_3) \end{aligned}$$

The local truncation error of RK-4 method is $O(h^5)$.

Example (7): Use RK-2 method to obtain the numerical solution $y(1.01)$ and $y(1.02)$ of the initial value problem

$$\frac{dy}{dx} = 1 + y^2 + x^3, \quad y(1) = -4$$

Solution :

$$h = 0.01, x_0 = 1, y_0 = -4, f(x, y) = 1 + y^2 + x^3$$

$$x_1 = 1.01, y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = -3.8254$$

$$k_1 = f(x_1, y_1) = (1 + y_1^2 + x_1^3) = 16.66$$

$$k_2 = f(x_1 + h, y_1 + k_1 h) = (1 + (y_1 + 0.1666)^2 + (x_1 + .01)^3) = 15.45$$

$$y_2 = y_1 + \frac{h}{2}(k_1 + k_2) = -3.8254 + \frac{0.01}{2}(16.66 + 15.45) = -3.6648$$

Example (8): Solve the I.V.P.

$$y' = x - y, \quad y(0) = 1$$

By using RK-4 method at the points $x=0.1, 0.2, 0.3, 0.4$

Solution:

$$f(x, y) = x - y, \quad x_0 = 0, \quad y_0 = 1, \quad h = 0.1$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\text{where } k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1)$$

$$k_3 = hf(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2)$$

$$k_4 = hf(x_i + h, y_i + k_3)$$

$$k_1 = hf(x_0, y_0) = 0.1(0 - 1) = -0.1$$

$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = 0.1f(0.05, 1 + \frac{1}{2}(-0.1)) = 0.1(0.05 - 0.95) = -0.09$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = -0.0905$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = -0.08095$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.909675$$

Also

$$y_2=0.837461$$

$$y_3=0.781636$$

$$y_4=0.7406405$$

Runge-Kutta Method Algorithm of order four

Input: n , x_0 , y_0 , $f(x,y)$

Step(1): compute $h = \frac{b-a}{n}$

Step(2): For $i=1,2,\dots,n$

Step(3): Find

$$k_1 = hf(x_{i-1}, y_{i-1})$$

$$k_2 = hf(x_{i-1} + \frac{1}{2}h, y_{i-1} + \frac{1}{2}k_1)$$

$$k_3 = hf(x_{i-1} + \frac{1}{2}h, y_{i-1} + \frac{1}{2}k_2)$$

$$k_4 = hf(x_{i-1} + h, y_{i-1} + k_3)$$

Step(4): Evaluate $y_i = y_{i-1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$x_i = x_{i-1} + h$$

Step(5): Print x_i and y_i for all i , then stop.