2) Finite Differences

There are three types of differences

1- Forward differences.
2- Backward differences.
3- Central differences

Let $y = f(x)$ be a function defined on $(n+1)$ points that is :-

$$x_0 , x_1 , \ldots , x_n$$

where
$$x_i = x_0 + ih$$
$$= x_{i-1} + h$$

اي ان الفرق بين قيم x المتتالية مقدار ثابت وقيمته h .

1- Forward Differences :-

Let $y_0 , y_1 , \ldots , y_n$ denote the set of values taken by the function $y = f(x)$ then

$$\Delta f(x) = f(x+h) - f(x) \qquad (\text{first forward differences})$$

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

$$\Delta f_i = f_{i+1} - f_i \quad , \quad \forall \; i = 0, 1, \ldots, n-1$$

$$\Delta^2 f(x_i) = \Delta(\Delta f(x_i)) \qquad (\text{second forward differences})$$
$$= \Delta f(x_{i+1}) - \Delta f(x_i)$$
$$= f(x_{i+2}) - f(x_{i+1}) - (f(x_{i+1}) - f(x_i))$$
$$= f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)$$

$$\Delta^2 f_i = f_{i+2} - 2f_{i+1} + f_i \,, \quad \forall\, i = 0, 1, \ldots, n-2$$

In general, the Kth forward differences at x_i is given by:

$$\Delta^k f_i = \Delta^{k-1}(\Delta f_i)$$
$$= \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i \,, \quad K = 1, 2, \ldots$$

Table of Forward differences (Δ)

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\ldots$	$\Delta^n f_i$
x_0	f_0					
		Δf_0				
			$\Delta^2 f_0$	$\Delta^3 f_0$	$\ldots$	
x_1	f_1					
		Δf_1				
			$\Delta^2 f_1$	$\Delta^3 f_1$		$\Delta^n f_0$
x_2	f_2					
		Δf_2		$\vdots$		
x_3	f_3		$\vdots$	$\Delta^3 f_{n-3}$		
$\vdots$	$\vdots$	$\vdots$				
			$\Delta^2 f_{n-2}$			
$\vdots$	$\vdots$	Δf_{n-1}				
x_n	f_n					

Ex: Construct a forward difference table for the following data.

x	1	2	3	4
$y = f(x)$	-1	-1	1	5

Sol:-

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
1	-1			
		$-1-(-1)=0$		
2	-1		2	
		$1-(-1)=2$		0
3	1			
			2	
4	5	$5-1=4$		

2 - Backward Differences:

Let $y = f(x)$ be a function given by the values $f_0, f_1, \dots, f_n$ which it takes for the equally spaced values $x_0, x_1, \dots, x_n$. Then

$$\nabla f(x) = f(x) - f(x-h) \qquad \text{(first backward diff.)}$$

$$\nabla f(x_i) = f(x_i) - f(x_i - h)$$

$$\nabla f_i = f_i - f_{i-1} \quad , \quad \forall i = n, n-1, \dots, 1$$

$$\nabla^2 f_i = f_i - 2f_{i-1} + f_{i-2} \quad , \quad \forall i = n, n-1, \dots, 2$$

(Second backward diff.)

In general;

$$\nabla^k f_i = \nabla^{k-1}(\nabla f_i) \quad , \quad \forall i = n, n-1, \dots, k$$

Table of backward differences (∇)

x_i	f_i	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$	$\nabla^4 f_i$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				