

Definiteness of Quadratic Form and eigenvalues

- (1) **محدد موجب (Positive definite)** $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ إذا كانت جميع الجذور المميزة لمصفوفة المحدد الموجب أكبر من الصفر
- (2) **محدد سالب (Negative definite)** $\lambda_1, \lambda_2, \dots, \lambda_n < 0$ إذا كانت جميع الجذور المميزة لمصفوفة المحدد السالب أصغر من الصفر
- (3) **محدد غير محدد (Indefinite)** $\lambda_1, \lambda_2, \dots, \lambda_n$ ≥ 0 إذا كانت جميع الجذور المميزة للمصفوفة أكبر أو تساوي الصفر
- (4) **محدد شبه سالب (Semi-definite)** $\lambda_1, \lambda_2, \dots, \lambda_n \leq 0$ إذا كانت جميع الجذور المميزة للمصفوفة أقل أو تساوي الصفر
- (5) **محدد غير محدد (Indefinite)** $\lambda_1, \lambda_2, \dots, \lambda_n$ ≥ 0 إذا كانت الجذور المميزة للمصفوفة أكبر أو تساوي الصفر

Ex:- Determine the definiteness of

$$Q(x) = -x_1^2 - 6x_1x_2 - 9x_2^2 - 2x_3^2$$

Soln:-

The symmetric matrix associated with the quadratic form Q is given

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 3 & 0 \\ 3 & -9-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda_1 = -2, \lambda_2 = -6, \lambda_3 = 0$$

Hence Q is negative Semidefinite

The Multivariate Normal Distribution

If $X \sim N(\mu, \sigma^2)$, Then the probability density function is:

$$f(x) = \frac{1}{\sqrt{2\pi} \sqrt{v(x)}} e^{-\frac{1}{2v(x)} (x - \mu)^2}$$

Since: $\mu_x = \mu$

$$v(x) = E[(X - \mu)^2] = \sigma^2$$

$$\therefore f(x) = \frac{1}{(\sqrt{2\pi})^n \sigma} e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2}$$

The correlation coefficient between X_1 & X_2 is

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sqrt{v(x_1) v(x_2)}}$$

$$\text{Let } \text{Cov}(X_1, X_2) = \sigma_{12}$$

$$\Rightarrow \rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \Rightarrow \sigma_{12} = \rho \sigma_1 \sigma_2$$

$$-1 \leq \rho \leq 1$$

If $\rho = 0$ That is mean X_1 & X_2 are independent
therefor $\sigma_{12} = 0$

Now, If $X_1, X_2, X_3, \dots, X_p$ are normally distribution
Then the joint probability density function is

$$f(x_1, x_2, \dots, x_p) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp \left[-\frac{1}{2} (\underline{x} - \underline{E}(\underline{x}))' \Sigma^{-1} (\underline{x} - \underline{E}(\underline{x})) \right] \quad \text{--- (1)}$$

such that

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \quad \underline{E}(\underline{x}) = \begin{bmatrix} E x_1 \\ E x_2 \\ \vdots \\ E x_p \end{bmatrix}$$

but $E(x_i) = \mu_i \quad i=1, 2, \dots, p \quad , \quad V(x_i) = \sigma_i^2$

$$\therefore E(\underline{x}) = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} = \underline{\mu}$$

Σ is variance - covariance matrix of $\underline{x}$ such that

$$\Sigma_{p \times p} = \text{var-cov}(\underline{x}) = E(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})'$$

$$= E \left\{ \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_p - \mu_p \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_p - \mu_p \end{pmatrix}' \right\}$$

$$= E \left\{ \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_p - \mu_p \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 & x_2 - \mu_2 & \dots & x_p - \mu_p \end{pmatrix} \right\}$$

$$= E \left\{ \begin{pmatrix} (x_1 - \mu_1)^2 & (x_1 - \mu_1)(x_2 - \mu_2) & \dots & (x_1 - \mu_1)(x_p - \mu_p) \\ (x_2 - \mu_2)(x_1 - \mu_1) & (x_2 - \mu_2)^2 & \dots & (x_2 - \mu_2)(x_p - \mu_p) \\ \vdots & \vdots & \ddots & \vdots \\ (x_p - \mu_p)(x_1 - \mu_1) & \dots & \dots & (x_p - \mu_p)^2 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} E(x_1 - \mu_1)^2 & E(x_1 - \mu_1)(x_2 - \mu_2) & \dots & E(x_1 - \mu_1)(x_p - \mu_p) \\ E(x_2 - \mu_2)(x_1 - \mu_1) & E(x_2 - \mu_2)^2 & \dots & E(x_2 - \mu_2)(x_p - \mu_p) \\ \vdots & \vdots & \ddots & \vdots \\ E(x_p - \mu_p)(x_1 - \mu_1) & \dots & \dots & E(x_p - \mu_p)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{12} & \sigma_2^2 & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \dots & \sigma_p^2 \end{pmatrix}$$

Therefore, The Variance-Covariance matrix is the matrix such that the main diagonal elements are the variance, and the other elements are the covariance, and Σ is positive definite matrix, i.e. $|\Sigma| > 0$

therefore, eq (1) becomes,

$$L(x_1, x_2, \dots, x_p) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (\underline{x} - \underline{\mu})' \Sigma^{-1} (\underline{x} - \underline{\mu}) \right\}$$

----- (2)