$$\therefore f(\underline{X}^{(1)} / \underline{X}^{(2)} = \underline{x}^{(2)}) \sim N_q[\underline{\mu}^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}^{(2)} - \underline{\mu}^{(2)}), \Sigma_{11.2}]$$

where: $\Sigma_{11.2} = \Sigma_{11} \cdot \Sigma_{12}$

$= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$

and

$$f(X^{(2)} / X^{(1)} = x^{(1)}) = \frac{1}{(2\pi)^{\frac{p-q}{2}} |\Sigma_{22}|^{1/2}} e^{-\frac{1}{2}[(x^{(2)} - \mu^{(2)})' \Sigma_{22}^{-1} (x^{(2)} - \mu^{(2)})]}$$

$$\therefore f(X^{(2)} / X^{(1)} = x^{(1)}) \sim N_{p-q}(\mu^{(2)}, \Sigma_{22})$$

Note:

① $E(X^{(1)} | X^{(2)} = x^{(2)}) = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$

is the linear function on $x^{(2)}$

i.e. The conditional expectation of $X^{(1)} | X^{2} = x^{2}$ is the linear function on $x^{(2)}$

التوقع الشرطي $X^{(1)}|X^{(2)}=x^{2}$ هو دالة خطية في $x^{(2)}$

② $V(X^{(1)} | X^{(2)} = x^{(2)}) = \Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}$

did not dependent on $X^{(2)}$

③ In the simple linear Regression model:

$Y/X = \alpha + BX + u$

$E(Y/X) = \alpha + BX$

$\therefore E(X^{(1)} / X^{(2)} = x^{(2)}) = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$

Therefore, This form is called the multivariate regression equation of $X^{(1)}$ on $X^{(2)}$, With the $\Sigma_{12}\Sigma_{22}^{-1}$ is called the multivariate regression coefficients of $X^{(1)}$ on $X^{(2)}$ and intercept مقطع $\mu^{(1)}$.

$(\Sigma_{12}\Sigma_{22}^{-1})$

(4) The element ij is regression coefficients matrix is indexed as $\beta_{ij\cdot q+1,q+2,\dots,p}$

العنصر (ij) في مصفوفة معاملات الانحدار يعبر عنه ب $\beta_{ij\cdot q+1,\dots,p}$

(5) The element ij in $\Sigma_{11.2}$ is indexed as

$\sigma_{ij\cdot q+1,q+2,\dots,p}$ which is called the partial covariance between the variate X_i, X_j. and $\sigma_{ii,q+1,\dots,p}$ is partial variance, then:-

العنصر في مصفوفة $\Sigma_{11.2}$ يعبر عنه ب $\sigma_{ij\cdot q+1,q+2,\dots,p}$ والذي
عن التباين المشترك الجزئي بين المتغيرين X_i, X_j و $\sigma_{ii,q+1,\dots,p}$
عن التباين الجزئي للعنصر X_i ..

$$\rho_{ij,q+1,\dots,p} = \frac{\sigma_{ij,q+1,q+2,\dots,p}}{\sqrt{\sigma_{ii,q+1,\dots,p}\cdot\sigma_{jj,q+1,\dots,p}}}$$

is the partial correlation coefficient between X_i, X_j holding $X_{q+1,q+2,\dots,p}$ fixed

Example (1):

If $\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, $\mu = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -4 \\ -1 & -4 & 8 \end{pmatrix}$

i.e: $\underline{X} \sim N_3 \left(\begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix}, \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -4 \\ -1 & -4 & 8 \end{pmatrix} \right)$

Find:-

1. $f(X^{(1)} \mid X^{(2)} = x^{(2)})$
2. $V(X^{(1)} \mid X^{(2)} = x^{(2)} = 50)$
3. $E(X^{(1)} \mid X^{(2)} = 4)$
4. vector of regression equation
5. $\sigma_{23.1}$ (6) $\rho_{23.1}$

such that $X^{(1)} = \begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$, $X^{(2)} = (X_1)$

Solution:

$$X^{(2)} = X_1 \quad , \mu^{(2)} = 9$$

(1) $f(X^{(1)} \mid X^{(2)} = x^{(1)}) = \dfrac{1}{(2\pi)^{q/2} |\Sigma_{11.2}|^{\frac{1}{2}}} * e^{-\frac{1}{2}[x^{(1)} - (\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x_2^{(2)} - \mu^{(2)})]}$

$$\Sigma_{11.2}^{-1} \left[x^{(1)} - (\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)}) \right]$$

$$\underline{X} = \begin{pmatrix} X_2 \\ X_3 \\ \hline X_1 \end{pmatrix}, \quad \mu = \begin{pmatrix} 2 \\ -1 \\ \hline 9 \end{pmatrix}, \quad \Sigma = \begin{array}{c} \\ x_2 \\ x_3 \\ x_1 \end{array} \begin{array}{ccc} X_2 & X_3 & X_1 \\ \left(\begin{array}{cc|c} 3 & -4 & 1 \\ -4 & 8 & -1 \\ \hline 1 & -1 & 2 \end{array}\right) \end{array}$$

$$\Sigma_{11} = \begin{pmatrix} 3 & -4 \\ -4 & 8 \end{pmatrix}, \quad \Sigma_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \Sigma_{21}^{t}$$

$$\Sigma_{22} = 2$$

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= \begin{pmatrix} 3 & -4 \\ -4 & 8 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -4 \\ -4 & 8 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2.5 & -3.5 \\ -3.5 & 7.5 \end{pmatrix}$$

$$|\Sigma_{11.2}| = \boxed{6.5}$$

$$\Sigma_{11.2}^{-1} = \begin{pmatrix} 1.154 & 0.538 \\ 0.538 & 0.385 \end{pmatrix}$$

$$\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{2} (X_1 - 9)$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0.5X_1 - 4.5 \\ -0.5X_1 + 4.5 \end{pmatrix} = \begin{pmatrix} 0.5X_1 - 2.5 \\ -0.5X_1 + 3.5 \end{pmatrix}$$

(10)

$$\therefore f(x^{(1)}/x^{(2)}=x^{(2)}) = \frac{1}{(2\pi)^{2/2}\sqrt{6.5}} e^{-\frac{1}{2}\left[\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0.5X_1 - 2.5 \\ -0.5X_1 + 3.5 \end{pmatrix}\right]'}$$

$$\begin{bmatrix} 1.154 & 0.538 \\ & 0.385 \end{bmatrix}\left[\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} - \begin{pmatrix} 0.5X_1 - 2.5 \\ -0.5X_1 + 3.5 \end{pmatrix}\right]$$

$$f(x^{(1)}/x^{(2)}=x^{(2)}) = \frac{1}{(2\pi)\sqrt{6.5}} e^{-\frac{1}{2}\left[1.154(X_2 - 0.5X_1 + 2.5)^2 + 0.385\left[X_3 - 0.5X_2 - 3.5\right]^2 + 2(0.538)(X_2 - 0.5X_1 + 2.5)(X_3 + 0.5X_1 - 3.5)\right]}$$

(2) $E(X^{(1)}/X^{(2)}=4) = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x^{(2)} - \mu^{(2)})$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{2}(4-9)$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -2.5 \\ 2.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}$$

(3) $V(X^{(1)}/X^{(2)}=x^{(2)}=5) = \Sigma_{11.2} = \begin{pmatrix} 2.5 & -3.5 \\ & 7.5 \end{pmatrix}$

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④ vector of regression equation :-

$$E\left(X^{(1)} \mid X^{(2)} = x^{(2)}\right) = \mu^{(1)} + \Sigma_{12}\,\Sigma_{22}^{-1}\left(\underset{\substack{\Downarrow \\ x_1}}{x^{(2)}} - 9\right)$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0.5\,x_1 - 4.5 \\ -0.5\,x_1 + 4.5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E(x_2 / x_1) \\ E(x_3 / x_1) \end{pmatrix} = \begin{pmatrix} -2.5 + 0.5\,x_1 \\ 3.5 - 0.5\,x_1 \end{pmatrix}$$

⑤ $\sigma_{ij \cdot q+1, \dots, p}$, i,j th the element of $\Sigma_{11.2}$

$$\Sigma_{11.2} = \begin{array}{c|cc} & X_2 & X_3 \\ \hline X_2 & 2.5 & -3.5 \\ X_3 & & 7.5 \end{array}$$

$$\therefore \ \sigma_{23.1} = -3.5$$

⑥ $$\rho_{ij \cdot q+1, \dots, p} = \frac{\sigma_{ij \cdot q+1, \dots, p}}{\sqrt{\sigma_{ii \cdot q+1, \dots, p}}\ \sqrt{\sigma_{jj \cdot q+1, \dots, p}}}$$

$$\rho_{23.1} = \frac{\sigma_{23.1}}{\sqrt{\sigma_{22.1}}\ \sqrt{\sigma_{33.1}}} = \frac{-3.5}{\sqrt{2.5}\ \sqrt{7.5}} = -0.81$$

.: الارتباط عكسي قوي

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H.W:

Let $\underline{X} \sim N_3\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}\right)$

المطلوب :

نفس ما هو مطلوب بالمثال السابق .