The Multivariate Normal Distribution

If $X \sim N(\mu, \sigma^2)$, Then the probability density function is:

$$f(x) = \frac{1}{\sqrt{2\pi}\ \sqrt{V(x)}}\, e^{-\frac{1}{2V(x)}(x - Ex)^2}$$

since: $Ex = \mu$

$V(x) = E(x - \mu)^2 = \sigma^2$

$$\therefore f(x) = \frac{1}{(2\pi)^{1/2}\ \sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma^2}\right)^2}$$

The Correlation Coefficient between X_1 & X_2 is

$$\rho = \frac{Cov(X_1, X_2)}{\sqrt{V(X_1)\, V(X_2)}}$$

Let $Cov(X_1, X_2) = \sigma_{12}$

$$\Rightarrow \rho = \frac{\sigma_{12}}{\sigma_1\ \sigma_2} \Longrightarrow \sigma_{12} = \rho\, \sigma_1\, \sigma_2$$

$$-1 \leq \rho \leq 1$$

If $\rho = 0$ That is mean X_1 & X_2 are independents therefor $\sigma_{12} = 0$

Now, if $X_1, X_2, X_3 \cdots X_p$ are normally distribution Then the joint probability density function is

$$f(x_1, x_2, \cdots x_p) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\underline{x} - E(\underline{x}))' \Sigma^{-1} (\underline{x} - E(\underline{x}))\right] \quad \cdots\cdots (1)$$

such that

$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \quad E(\underline{x}) = \begin{bmatrix} E x_1 \\ E x_2 \\ \vdots \\ E x_p \end{bmatrix}$$

but $E(x_i) = \mu_i \quad i = 1, 2 \cdots P \quad , V(x_i) = \sigma_i^2$

$$\therefore E(\underline{X}) = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} = \underline{\mu}$$

Σ is variance - covariance matrix of X such that

$$\Sigma_{p \times p} = v\text{-}cov(\underline{X}) = E(\underline{X} - \underline{\mu})(\underline{X} - \mu)'$$

$$= E \left\{ \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_p - \mu_p \end{pmatrix} - \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_p - \mu_p \end{pmatrix}' \right\}$$

$$= E\left\{\begin{pmatrix} X_1-\mu_1 \\ X_2-\mu_2 \\ \vdots \\ X_p-\mu_p \end{pmatrix}\begin{pmatrix} X_1-\mu_1 & X_2-\mu_2 & \cdots & X_p-\mu_p \end{pmatrix}\right\}$$

$$= E\begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) & \cdots & (X_1-\mu_1)(X_p-\mu_p) \\ (X_2-\mu_2)(X_1-\mu_1) & (X_2-\mu_2)^2 & \cdots & (X_2-\mu_2)(X_p-\mu_p) \\ \vdots & & & \\ (X_p-\mu_p)(X_1-\mu_1) & \cdots & \cdots & (X_p-\mu_p)^2 \end{bmatrix}$$

$$= \begin{bmatrix} E(X_1-\mu_1)^2 & E(X_1-\mu_1)(X_2-\mu_2) & \cdots & E(X_1-\mu_1)(X_p-\mu_p) \\ E(X_2-\mu_2)(X_1-\mu_1) & E(X_2-\mu_2)^2 & \cdots & E(X_2-\mu_2)(X_p-\mu_p) \\ \vdots & & & \\ \vdots & & & E(X_p-\mu_p)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ & \sigma_2^2 & \cdots & \sigma_{2p} \\ & & \ddots & \\ & & & \sigma_p^2 \end{bmatrix}$$

Therfor, The variance-Covariance matrix is the matrix such That the main diagonal elements are the variance, and the other elements are the covariance, and Σ is positive definite matrix, i.e $|\Sigma| > 0$

Therefore, eq (1) becomes,

$$f(x_1, x_2, \ldots x_p) = \frac{1}{(2\pi)^{p/2} \, |\Sigma|^{\frac{1}{2}}} \exp\left[\frac{1}{2} (\underline{x} - \underline{\mu})' \, \Sigma^{-1} (\underline{x} - \underline{\mu})\right\} \quad \text{--- } (2)$$