Numerical methods:

(1) Bisection Method:

Suppose a continuous function f defined on the interval [a,b] is given with f(a) and f(b) of opposite sign (i.e. $f(a)\times f(b)<0$). Then by intermediate value theorem (If $f\in C[a,b]$ and k is any number between f(a) and f(b), then there exists $c\in(a,b)$ for which f(c)=k) there exists a point $c\in(a,b)$ such that f(c)=0. If we choose the midpoint $c=\frac{a+b}{2}$, then three possibilities aries:

$$\text{If } f(a)\times f(c) \begin{cases} <0 & \text{there is a root between a, c} \Rightarrow d=\frac{a+c}{2} \\ >0 & \text{there is a root between b, c} \Rightarrow d=\frac{b+c}{2} \\ =0 & \text{c is exact root ((Stop)).} \end{cases}$$

We stop iteration if the interval width is as small as desired i.e. $|c_i-c_{i+1}|<\varepsilon$ for any i.

<u>Example (5)</u>:

Find an approximate root of the equation $xsin(x)-1=0$ in the interval [0,2] by using Bisection method.

<u>Solution</u>: It is possible to use bisection method because f(x) is continuous on [0,2] and f(a)=f(0)=-1; f(b)=f(2)=0.81859 $\Rightarrow$ $f(a)\times f(b)<0$.

$\Rightarrow c_1=\frac{a+b}{2}=\frac{0+2}{2}=1$

$f(c_1)=f(1)=-0.158529$

$\Rightarrow f(c_1)\times f(a)>0 \Rightarrow$ there is a root between c_1 and b $\Rightarrow$ $[c_1,b]$

$\Rightarrow c_2=\frac{c_1+b}{2}=\frac{1+2}{2}=1.5$

$f(c_2)=f(1.5)=0.496242$

$\Rightarrow f(c_2)\times f(c_1)<0 \Rightarrow$ there is a root between c_1 and c_2 $\Rightarrow$ $[c_1,c_2]$

$\Rightarrow c_3=\frac{c_1+c_2}{2}=\frac{1+1.5}{2}=1.25$

$f(c_3)=f(1.25)= 0.18623$

$\Rightarrow f(c_3)\times f(c_1) <0 \Rightarrow$ there is a root between c_1 and c_3 $\Rightarrow$ $[c_1,c_3]$

$\Rightarrow c_4=\frac{c_1+c_3}{2}=1.125$

$\vdots$

Stop iteration with c=1.114157141

Example (6):

Find an approximate root of $f(x)=x^2-2$ in the interval [1,2] by using Bisection method with error $\leq \varepsilon = 10^{-4}$.

Solution:

It is possible to use bisection method because f(x) is continuous on [1,2] and $f(a)=f(1)=-1$; $f(b)=f(2)=2$

$\Rightarrow f(a)\times f(b)=-2<0.$

$c_1=\frac{a+b}{2}=\frac{1+2}{2}=1.5$, $|c_1-a|=0.5>\varepsilon$.

Find c_2:

$f(c_1)=0.25 \Rightarrow f(c_1)\times f(a)<0 \Rightarrow$ there is a root between c_1 and a $\Rightarrow [a,c_1]$

$\Rightarrow c_2=\frac{a+c_1}{2}=1.25$

$|c_1-c_2|=0.25>\varepsilon$.

Find c_3:

$f(c_2)=-0.437 \Rightarrow f(c_2)\times f(c_1)<0 \Rightarrow$ there is a root between c_1 and $c_2 \Rightarrow [c_2, c_1]$

$\Rightarrow c_3=\frac{c_2+c_1}{2}=1.375$

$\vdots$

Stop iteration if $|c_i-c_{i+1}| \leq\varepsilon$ for any i=1, 2, ….

Theorem:

Let $f \in C[a,b]$ and suppose $f(a) \times f(b) < 0$. The bisection method generates a sequence $\{c_n\}$ approximating to P a zero of f ($f(P)=0$) with the property $|c_n - P| \le \frac{b-a}{2^n};\ n \ge 1.$

Notes:

1. **1-** The rate of convergence is linear.
2. **2-** Since $|c_n - P| \le \frac{b-a}{2^n};\ n \ge 1,$ then the sequence $\{c_n\}$ converges to P with rate of convergence $O(1/2^n)$ that is $c_n = P + O(1/2^n)$.
3. **3-** We can determine approximately how many iterations are necessary to solve f(x) with error $\le \varepsilon$ over [a,b]. i.e.:

We must find an integer n that will satisfy

$$|c_n - P| \le \frac{b-a}{2^n} \le \varepsilon$$

$$\Rightarrow 2^n \ge \frac{b-a}{\varepsilon} \Rightarrow n\ln(2) \ge \ln\left(\frac{b-a}{\varepsilon}\right)$$

$$\Rightarrow n \ge \frac{\ln(b-a) - \ln(\varepsilon)}{\ln(2)}.$$

For example, if $f(x) = x\sin(x) - 1 = 0$, $\varepsilon = 10^{-5}$ and [0,2], then

$$n \ge \frac{\ln(2-0) - \ln((10^{-5})}{\ln(2)} = \frac{0.69315 - (-11.51293)}{0.69315} \approx 17.6072 \Rightarrow$$ n=18(number of iterations).

<u>**Home work:**</u>

Find an approximate root of $f(x)=x\ln(x)-1=0$ in the interval [1,2] by using Bisection method with error $\le \varepsilon = 10^{-3}$.

Answer the root is $c_{11}=1.762953125$ with error $\le \varepsilon = 10^{-3}$.

Algorithm (Bisection method)

Input: a, b, ε, f(x)

Step(1): If f(a)×f(b) >0 then stop (does not exist root).

Step(2): Set c= $\frac{a+b}{2}$, and find f(c).

Step(3): If f(a)×f(c) <0 then b=c

Step(4): If f(a)×f(c) >0 then a=c

Step(5): If $|b-a| \geq \varepsilon$ or $|f(c)| \geq \varepsilon$ then go to step(2).

Step(6): Print c